Fourier Conjugate for shock detection

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Qing Cheng , School of Mathematical Sciences , Xiamen University .

Introduction

The solution of a hyperbolic equation may have many discontinuities, especially for problems containing both shocks and complicated smooth solution structures, such as, compressible turbulence simulations and aeroacoustics. The key idea in this research is to detect such discontinuities using Fourier conjugate Method.

Fourier Conjugate

Consider a trigonometric series of the form:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{N}\left(a_n\cos(nx)+b_n\sin(nx)\right). \quad (1)$$

The Fourier conjugate of $f(x)$ is

$$\hat{f}(x)=\sum_{n=1}^{N}\left(a_n\sin(nx)-b_n\cos(nx)\right), \quad (2)$$

and its frist derivative is

$$\hat{f}^{'}(x)=\sum_{n=1}^{N}\left(na_n\cos(nx)+nb_n\sin(nx)\right). \quad (3)$$

The low frequency oscillations can be removed from the Fourier modes by applying a high-pass filter to $\hat{f}(x)$.

$$\hat{f}^{\sigma}(x)=\sum_{n=1}^{N}\sigma\left(\frac{k}{n}\right)\left(a_n\sin(nx)-b_n\cos(nx)\right). \quad (4)$$

where the high-pass filter is

$$\sigma\left(\frac{k}{N}\right)=1-e^{(-\alpha(k/N))^{\gamma}}. \quad (5)$$

$\alpha=-\ln(\epsilon)$, ϵ is machine zero, N is the number of Fourier modes and γ is the order of filter.
An important property of Fourier conjugate of $f(x)$ is that it converges to the jump of $f(x)$ at each x.

$$-\frac{\pi}{\log(N)}\hat{f}\rightarrow[f](x)\delta_{\xi}(x), thatis \quad (6)$$

$$[f](x)\delta_{\xi}(x)=\begin{cases}[f(x)](\xi) & x=\xi\\ 0 & otherwise\end{cases}, where \quad (7)$$

$\delta_{\xi}(x)$ is the Kronecker symbol and $[f]=f^{+}-f^{-}$.

Square Function Analysis

Consider the square function and its Fourier conjugate

$$f(x)=\begin{cases}1 & |x|\le 1\\ 0 & 1\le|x|\le\pi\end{cases}. \quad (8)$$

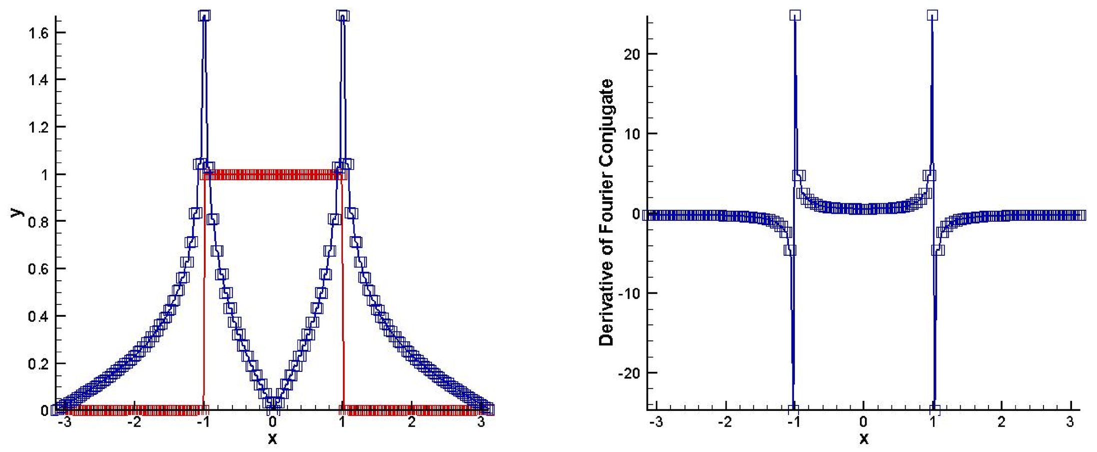

Figure 1: (Left) the absolute value and (Right) the first derivative of Fourier conjugate for the square function with $N=200$ points.

The Fourier conjugate of $f(x)$ converges to the jump location of $f(x)$ in fig. 1. In the smooth region, the first derivative of Fourier conjugate will be very small, but at the points around the discontinuities it will be very large.

Sod problem analysis

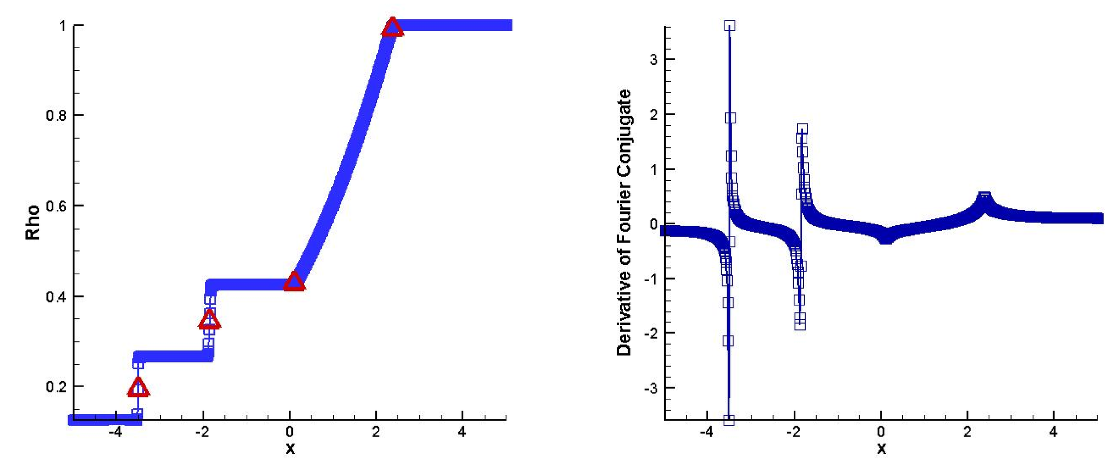

Figure 2: (Left) the density (ρ), (right) the first derivative of Fourier conjugate of ρ of Sod problem.

In fig.2, the two largest discontinuities at $x=-3.5$ and $x=-1.9$ can be identified by setting a criterion to cut off the first derivative of Fourier conjugate of ρ, where the criterion is

$$TVD=\frac{\sum_{i=1}^{N}|y_i-y_{i-1}|}{N}. \quad (9)$$

However discontinuities in the first derivative of density ρ at $x=0.1$ and $x=2.4$ are difficult to identify.

Sod problem Analysis

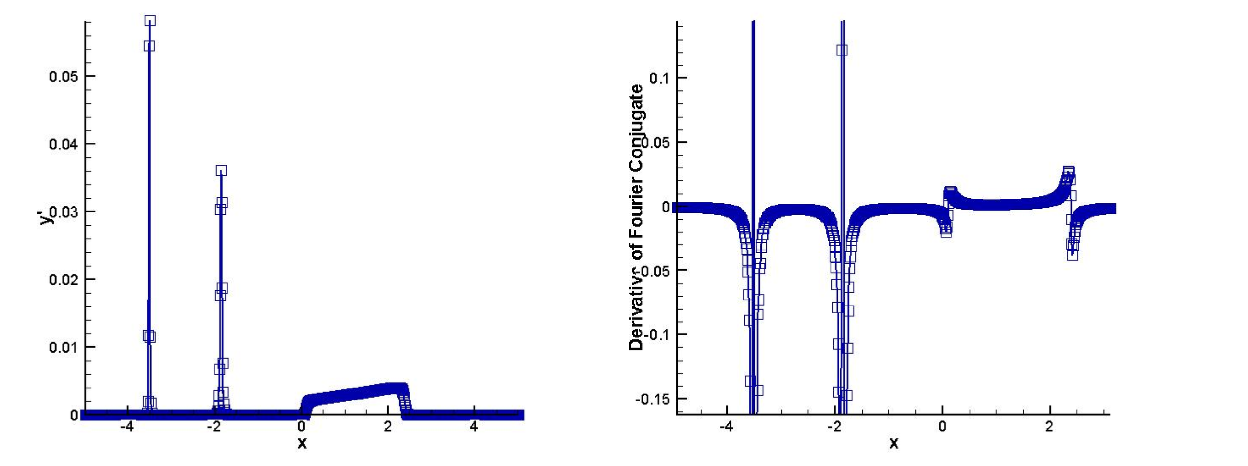

Figure 3: (left) backward difference of density ρ function $\tilde{f}$, and (right) the first derivative of Fourier conjugate of $\tilde{f}$.

In fig.3 the discontinuities in the first derivative of density ρ at $x=0.1$ and $x=2.4$ are identified by setting the TVD criterion to cut off the first derivative of $\tilde{f}$ while ignoring the TVD around the points at $x=-3.5$ and $x=-1.9$.

Shu-Osher problem

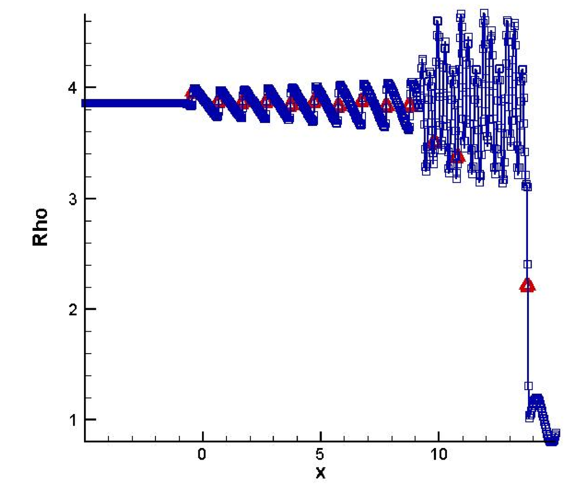

Figure 4: $N=800$, red triangles with a triangular are the shock locations as detected by Fourier conjugate .

In fig.4, in Shu-Osher problem, the density ρ contains discontinuities and high frequency waves, and the discontinuities need to be separated from the high frequency waves. In fig.5 (left), the TVD criterion is difficult to insulate discontinuities from the high frequency waves. So the high-pass filter is used to removed the high frequency waves.

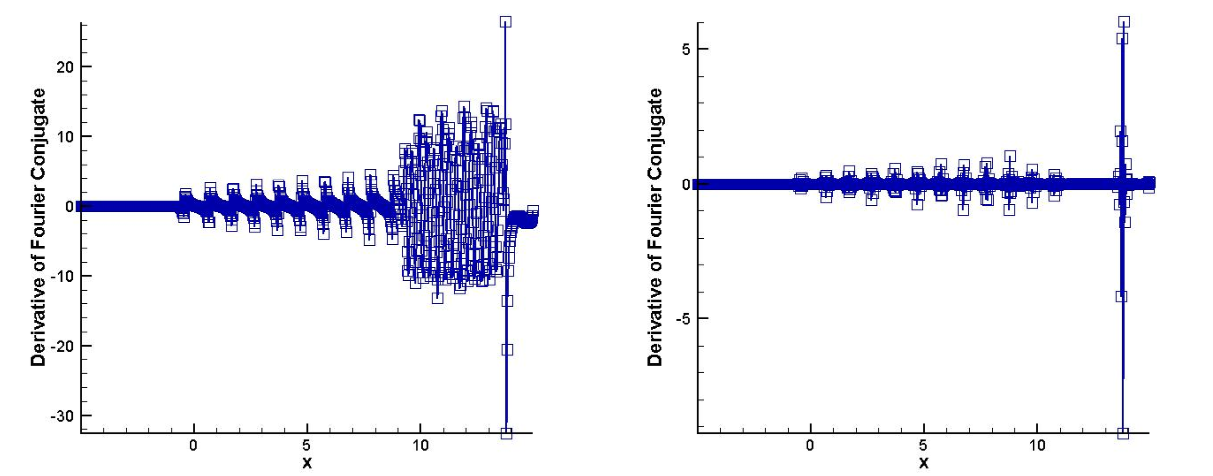

Figure 5: (left) no filter (right) filter order=8 for the first derivative of Fourier conjugate with $N=800$ points.

In fig.5 (right), after adding the high-pass filter to the first derivative of Fourier conjugate of ρ, the high frequency part is removed. The discontinuities of density ρ are identified from the high frequency waves by setting the criterion to cut off the TVD of the first derivative of Fourier conjugate of ρ.

References

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Future Work

- Detect the discontinuities more accurately from the high frequency wave .
- Find fast algorithm for the Fourier conjugate method.
- Hybridization of the WENO scheme with Spectral Method for 1D Burgers' equation and Euler equation .
- Hybridization of the WENO scheme with Spectral Method for 2D and 3D Burgers' equation and Euler equation .

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