High Order Multi-Resolution Analysis

Wai Sun Don

BROWN

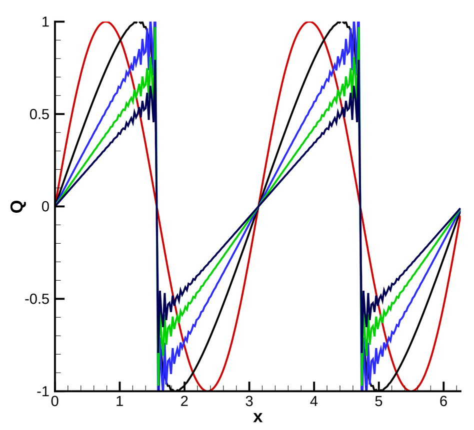
OCEAN UNIVERSITY OF CHINA

Introduction

Consider the hyperbolic conservation laws

$$\frac{\partial \mathbf{Q}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{Q}) = 0. \qquad (1)$$

It is well known that singularities (shocks) can appear even with a smooth initial condition. Non-physical Gibbs oscillations will be generated if (1) is solved by a *linear* scheme (Fourier collocation method) as shown in the temporal evolution of the invisicid Burgers equation.



Hybrid scheme

High order WENO schemes (see [2, 3] and references contained therein) are a popular *nonlinear* scheme for solving (1). The WENO reconstruction procedure avoids interpolation across nonsmooth stencils, essentially localizing and minimizing the Gibbs oscillations near discontinuities, while resolving fine scale structures efficiently. However, they are expensive (flux-splitting, Roe eigensystem, characteristic forward and backward projections, smoothness indicators, and nonlinear weights), dissipative and dispersive, and complicated to implement.

Hybrid Scheme : To alleviate some of these difficulties, one should employ a linear scheme in the *smooth* regions of the solution.

High Order linear schemes can be

- Central scheme: local, general, simple, non-dissipative, explicit filtering, dispersive and high formal order.
- Compact scheme: global with boundary closure, relatively simple, non-dissipative, explicit filtering, dispersive and high formal order and resolution.

Question : How to determine the smoothness with high order of a function on an equi-distant grid?

Answer : High order Multi-resolution analysis (MR) by Harten [1, 4].

Multi-Resolution Analysis

Consider two nested dyadic grids $G^k = \{x_i^k = i\Delta x_k, \quad i = 0, \ldots, N_k\}$, $k = 0, 1$ where $\Delta x_1 = 2\Delta x_0$, $N_1 = N_0/2$ and the cell average of u:

$$\bar{u}_i^k = \frac{1}{\Delta x_k} \int_{x_{i-1}^k}^{x_i^k} u(x)dx, \quad k = 0, 1. \qquad (2)$$

Let $\tilde{u}_{2i-1}^k$ be the approximation to $\bar{u}_{2i-1}^k$ by a unique $2s$ degree polynomial that interpolates $\bar{u}_{i+l}^k, |l| \le s$ at x_{i+l}^k, where $q = 2s+1$ is the order of approximation. The multi-resolution (MR) coefficients, $d_i = |\bar{u}_{2i-1}^0 - \tilde{u}_{2i-1}^0|$ at x_i, has the property that if $u(x)$ is a C^{p-1} function, then

$$d_i \approx \begin{cases} [u_i^{(p)}]\Delta x_1^p & p \le q \\ u_i^{(q)} \Delta x_1^q & p > q \end{cases}, \qquad (3)$$

where $[\cdot]$ and $(\cdot)$ denote the jump and derivatives of the function, respectively. The d_i measures how close the data at the finer grid $\{x_i^0\}$ can be interpolated by the data at the coarser grid $\{x_i^1\}$. From (3), one has

$$d_{2i}^0 \approx 2^{-\min\{p,q\}} d_i, \qquad (4)$$

which implies that that away from discontinuities, d_i becomes smaller as the grid is refined; at discontinuities, they remain essentially the same size, independent of the order q.

MR Coefficients

Consider an equidistant grid $\{x_j = j\Delta x, j = -m, \ldots, N+M\}$ with grid spacing Δx, where m and M are number of ghost points as given in the Table.

N	n_{MR}	m	M
odd	odd	$n_{MR}+1$	$n_{MR}+1$
even	odd	$n_{MR}+1$	n_{MR}
odd	even	n_{MR}	$n_{MR}+2$
even	even	n_{MR}	$n_{MR}+1$

The average values of $f(x)$ at x_i are

$$\bar{f}_i = \frac{f_{2i} + f_{2i+1}}{2}, \; i = -\frac{m}{2}, \ldots, \frac{N+M-1}{2}. \qquad (5)$$

The $k = n_{MR}$ degree polynomial $P_k(x)$ is constructed, for even i, with $l = m/2$ and $L = l - \mod(k, 2)$, as

$$P_k(x_i) = \sum_{r=-l}^{L} \alpha_r \bar{f}_{i+r} = f(x_i) + O(\Delta x^{k+1}). \qquad (6)$$

If $\mod(k,2) = 0$, $\alpha_{-r} = \alpha_r$. For odd i, one replaces α_r with $\alpha_{-(r+1)}$.

MR Coefficients

The α_k are computed by requiring $P_k(x)$ to be equal to each of the first $k+1$ monomials $f(x) = 1, x, x^2, \ldots, x^k$ and evaluated at $x = 0$. The $\bar{f}_i$ are evaluated for $i = -l, \ldots, L$.
This procedure yields a system of equations, $\mathbf{A}\alpha = \mathbf{b}$, where, with $\xi = -2l$ and $\eta = 2L$,

$$\mathbf{A} = \begin{pmatrix} 1 & \ldots & 1 \\ \xi + (\xi+1) & \ldots & \eta + (\eta+1) \\ \vdots & \vdots & \vdots \\ \xi^k + (\xi+1)^k & \ldots & \eta^k + (\eta+1)^k \end{pmatrix}, \qquad (7)$$

$$\vec{\alpha} = \begin{pmatrix} \alpha_{-l} \\ \alpha_{-l+1} \\ \vdots \\ \alpha_L \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad (8)$$

and $\mathbf{A}$ is a $(L+l+1) \times (L+l+1)$ matrix.
Using (6), the k order MR coefficients d_i becomes

$$d_i = f_i - P_k(x_i) \quad i = 0, \ldots, N. \qquad (9)$$

For example, for the $k = n_{MR} = 3$ degree polynomial $P_k(x)$, such that

$$\begin{aligned} P_k(x_0) &= f(x_0) + O(\Delta x^{k+1}), \\ P_k(x_1) &= f(x_1) + O(\Delta x^{k+1}), \end{aligned}$$

(6) requires the coefficients vector $\vec{\alpha}$ to satisfy

$$\alpha_{-2}\bar{f}_{-2} + \alpha_{-1}\bar{f}_{-1} + \alpha_0 \bar{f}_0 + \alpha_1 \bar{f}_1 = f(x_0) + O(\Delta x^4)$$

which, from (7), yields

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -7 & -3 & 1 & 5 \\ 25 & 5 & 1 & 13 \\ -91 & -9 & 1 & 35 \end{pmatrix}, \quad \vec{\alpha} = \frac{1}{64}\begin{pmatrix} -3 \\ 17 \\ 55 \\ -5 \end{pmatrix}.$$

MR Flag

Using the MR coefficients d_i at x_i,

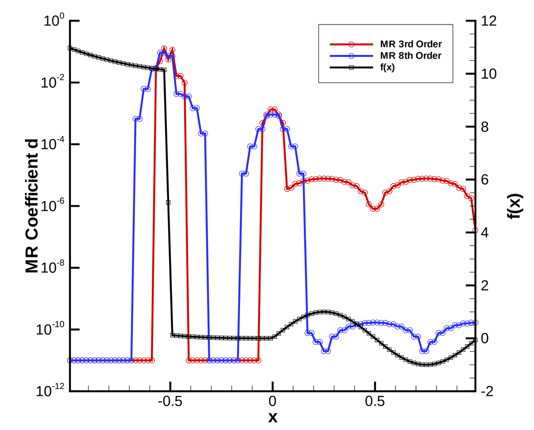
$$\text{Flag}_i = \begin{cases} 1, & |d_i| > \epsilon_{MR} \quad \text{(Non-Smooth)} \\ 0, & \text{otherwise} \qquad \text{(Smooth)} \end{cases}. \qquad (10)$$

where ϵ_{MR} (typically 1×10^{-3}) is a user tunable parameter.
A buffer zone is created around x_i if $\text{Flag}_i = 1$ to ensure a smooth transition between the smooth and non-smooth stencils.

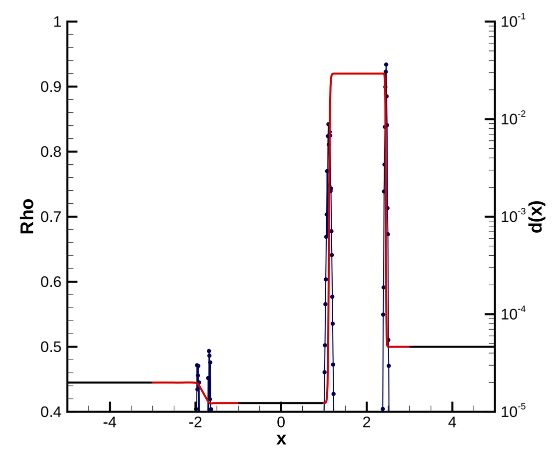
Examples

The piecewise analytic function : $n_{MR} = 3, 8$

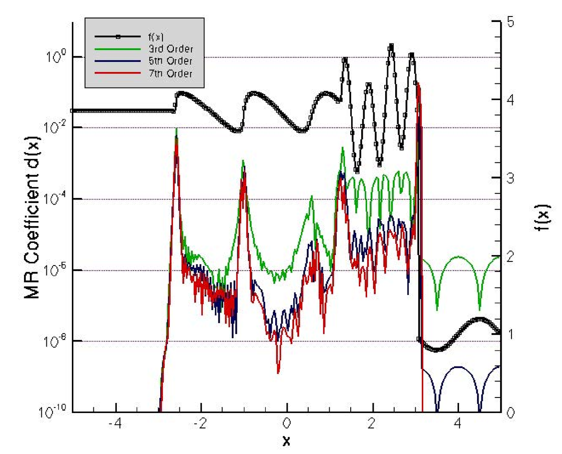
$$f(x) = \begin{cases} 10 + x^3 & -1 \le x < -0.5 \\ x^3 & -0.5 \le x < 0 \\ \sin(2\pi x) & 0 \le x \le 1 \end{cases}. \qquad (11)$$



Lax Problem
$n_{MR} = 5$

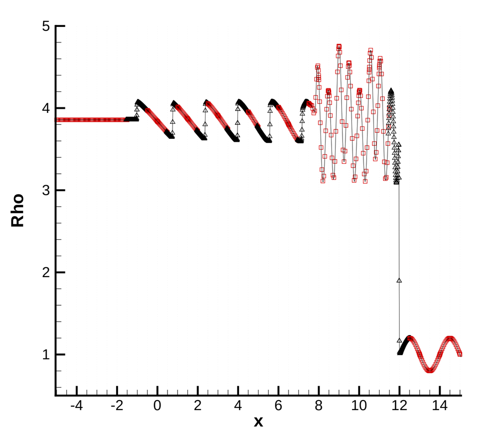


Shock-Density Wave
$n_{MR} = 3, 5, 7$

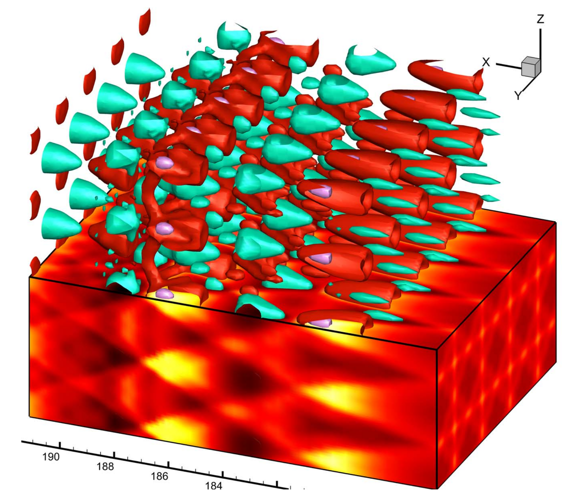


Hybrid Scheme

Shock-Density Wave



3D Stable Detonation



References

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[3] R. Borges, M. Carmona, B. Costa & W. S. Don, *An improved weighted essentially non-oscillatory scheme for hyperbolic conservation laws*, JCP **227** (2008)

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