Overestimated Quasi-conservative Formulation of Characteristic Finite-difference WENO Scheme for Multicomponent Compressible Fluid

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Introduction

The overestimated quasi-conservative form of the characteristic-interpolation-based finite-difference weighted essentially non-oscillatory(WENO) scheme, which maintains the equilibriums of velocity, pressure and temperature, is implemented to simulate compressible multicomponent flow fields. The WENO scheme is written in the split form that has the consistent and dissipation parts of the numerical flux. The dissipation part of the numerical flux is in the conservative form to maintain the conservation of conservative variables.

Overestimated quasi-conservative form

The overestimated quasi-conservative form of the governing equation is presented as follows:

$$\mathbf{Q}=\begin{bmatrix}\rho\\ \rho u\\ \rho e\\ \rho Y_1\\ \frac{1}{\gamma_p-1}\end{bmatrix},\quad \mathbf{M}=\begin{bmatrix}1&&&&\\&1&&&\\&&1&&\\&&&1&\\&&&&u\end{bmatrix},\quad \mathbf{F}=\begin{bmatrix}\rho u\\ \rho u^2+p\\ u(\rho e+p)\\ \rho u Y_1\\ \frac{1}{\gamma_p-1}\end{bmatrix},$$

$$\frac{\partial \mathbf{Q}}{\partial t}+\mathbf{M}\frac{\partial \mathbf{F}}{\partial x}=0.$$

Because we cannot define the flux Jacobian due to its quasi-conservative forms, the eigensystem of the following equation is considered instead:

$$\frac{\partial \mathbf{Q}}{\partial t}+\mathbf{B}\frac{\partial \mathbf{Q}}{\partial x}=0,\ \mathbf{B}=\mathbf{M}\frac{\partial \mathbf{F}}{\partial \mathbf{Q}},\mathbf{B}=\mathbf{R}\mathbf{\Lambda}\mathbf{L},$$

where $\mathbf{L}$ and $\mathbf{R}$ are the left and right eigenvectors of $\mathbf{B}$. The characteristic upwind fluxes are calculated by the Lax-Friedrichs flux splitting method:

$$f^{\pm,m}_{j+k}=\frac{1}{2}\mathbf{l_m}(\mathbf{F^M_{j+k,j+1/2}}\pm\lambda_m\mathbf{Q_{j+k}}),$$

where $\mathbf{l_m}$ is calculated by the Roe-averaged or algebraically averaged quantities of those on the nearest two points j and $j+1$. Here,

$$\lambda_m=\kappa\max(|\lambda_{ave,m}|,|\lambda_{j-2,m}|,...,|\lambda_{j+2,m}|),$$

and κ is great than 1.

$$\mathbf{F}^M_{j+k,j+1/2}=\begin{bmatrix}\rho_{j+k}u_{j+k}\\ \rho_{j+k}u^2_{j+k}+p_{j+k}\\ u_{j+k}(\rho_{j+k}e_{j+k}+p_{j+k})\\ \rho_{j+k}u_{j+k}Y_{1,j+k}\\ \tilde{u}_{j+1/2}(\frac{1}{\gamma_p-1})_{j+k}\end{bmatrix}.$$

The flux for the specific heat ratio includes constant $\tilde{u}_{j+1/2}$ in the stencil (i.e. $j-2$ to $j+2$ for the right direction wave of the fifth-order WENO scheme).

The split form of the WENO scheme

The standard WENO scheme is:

$$f^{W+,m}_{j+1/2}=\sum\omega^{+,m}_k f^{k+,m}_{j+1/2},$$

where

$$f^{0+,m}_{j+1/2}=\frac{1}{3}f^{+,m}_{j-2}-\frac{7}{6}f^{+,m}_{j-1}+\frac{11}{6}f^{+,m}_{j},$$

$$f^{1+,m}_{j+1/2}=-\frac{1}{6}f^{+,m}_{j-1}+\frac{5}{6}f^{+,m}_{j}+\frac{1}{3}f^{+,m}_{j+1},$$

$$f^{2+,m}_{j+1/2}=\frac{1}{3}f^{+,m}_{j}+\frac{5}{6}f^{+,m}_{j+1}-\frac{1}{6}f^{+,m}_{j+2}.$$

d_k are the *linear weights*, $d_0=\frac{1}{10},\ d_1=\frac{6}{10},\ d_2=\frac{3}{10}$.

$$\omega^{+,m}_k=\frac{\alpha^{+,m}_k}{\sum^2_{s=0}\alpha^{+,m}_s},\ k=0,1,2,$$

$$\alpha^{JS+,m}_k=\frac{d_k}{(\beta^{+,m}_k+\epsilon)^p}\ \text{or}\ \alpha^{Z+,m}_k=d_k\left(1+\left(\frac{\tau_5}{\beta^{+,m}_k+\epsilon}\right)^p\right),$$

where $\beta^{+,m}_k$ are *smoothness indicators* of the stencil S_k, which measures the smoothness of $f^{+,m}$ in stencil S_k. p is the power parameter and ϵ is the parameter to avoid the denominator to be zero. Then, the numerical flux in the physical fields is obtained:

$$\mathbf{F}^W_{j+1/2}=\sum_m\mathbf{r_m}(f^{W+,m}_{j+1/2}+f^{W-,m}_{j+1/2}).$$

Finally, the spatial derivatives are as follows:

$$\mathbf{M}\frac{\partial \mathbf{F}}{\partial x}\approx\frac{1}{\Delta x}(\mathbf{F}^W_{j+1/2}-\mathbf{F}^W_{j-1/2}).$$

The split form of the WENO scheme is:

$$\mathbf{M}_j\left(\frac{\partial \mathbf{F}}{\partial x}\right)_j\approx\mathbf{M}_j\frac{(\mathbf{F}^C_{j+1/2}-\mathbf{F}^C_{j-1/2})}{\Delta x}+\frac{(\mathbf{F}^D_{j+1/2}-\mathbf{F}^D_{j-1/2})}{\Delta x}.$$

$$\mathbf{F}^C_{j+1/2}=\frac{\mathbf{F}_{j-2}-8\mathbf{F}_{j-1}+37\mathbf{F}_j+37\mathbf{F}_{j+1}-8\mathbf{F}_{j+2}+\mathbf{F}_{j+3}}{60}.$$

$$\mathbf{F}^D_{j+1/2}=\sum_m\mathbf{r}_m(f^{D+,m}_{j+1/2}+f^{D-,m}_{j+1/2}).$$

$$\begin{aligned}f^{D\pm,m}_{j+1/2}&=\mathcal{W}^{m\pm}[f^{\pm,m}]=\mathcal{W}[f^{\pm,m},\omega^{\pm,m}_k]\\&=\mp\frac{1}{60}\left((20\omega^{\pm,m}_0-1)f'''^{0\pm,m}_{j+1/2}\right.\\&\quad\left.-(10(\omega^{\pm,m}_0+\omega^{\pm,m}_1)-5)f'''^{1\pm,m}_{j+1/2}+f'''^{2\pm,m}_{j+1/2}\right).\end{aligned}$$

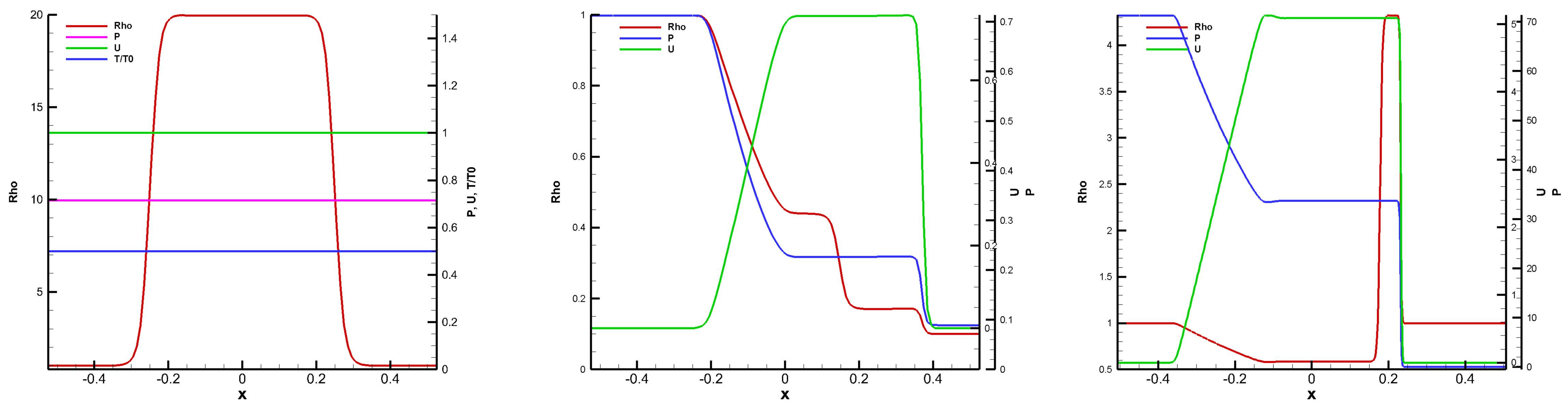
$$f'''^{k+,m}_{j+1/2}=-f^{+,m}_{j-2+k}+3f^{+,m}_{j-1+k}-3f^{+,m}_{j+k}+f^{+,m}_{j+1+k},$$

$$f'''^{k-,m}_{j+1/2}=-f^{-,m}_{j-k}+3f^{-,m}_{j+1-k}-3f^{-,m}_{j+2-k}+f^{-,m}_{j+3-k}.$$

To maintain the equilibrium of temperature, the weights for the second and fourth characteristic fluxes must be common by using the common smooth indicator defined as follows:

$$\beta^{2,4}_k=\beta^2_k+\beta^4_k,\ k=0,1,2$$

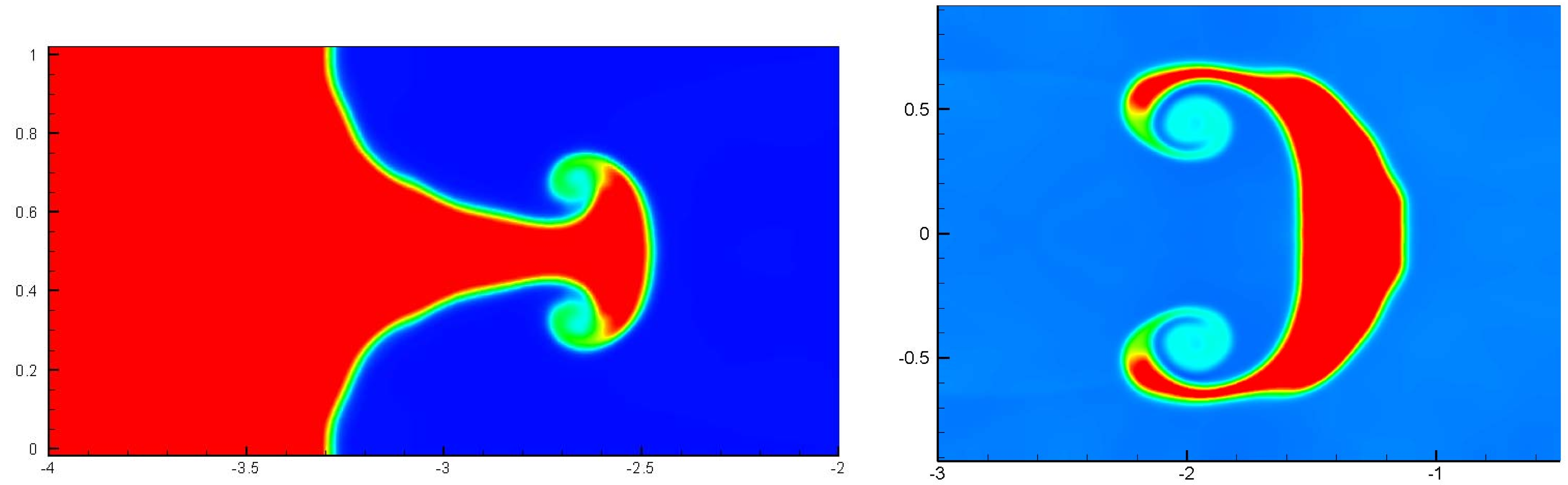
1D problem



The above three figures are results of moving material interface problem($t=1$), weak shock problem($t=0.2$) and stiff shock problem($t=0.035$).

The equilibriums of velocity, pressure and temperature for the first problem are confirmed to be maintained at machine zero level by using the overestimated quasi-conservative formulation. The other problems can maintain the equilibriums of velocity and pressure at density discontinuity by using OQC formulation.

2D problem



The above figures are density distributions of 2D Richtmyer-Meshkov instability problem ($t=8.25$, $\Delta x=\Delta y=\frac{1}{128}$) and shock-bubble interaction problem ($t=7.337$, $\Delta x=\Delta y=\frac{1}{100}$).

In 2D examples, OQC-WENO5 can maintain a smooth interface shape. This scheme can capture the interfaces without spurious oscillation.

Future Work

We plan to research on the overestimated quasi-conservative form of the alternative weighted essentially non-oscillatory (AWENO) scheme, which is required to maintain the equilibriums of velocity, pressure and temperature.

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References

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