Alternative WENO finite difference scheme with increasingly higher order accuracy

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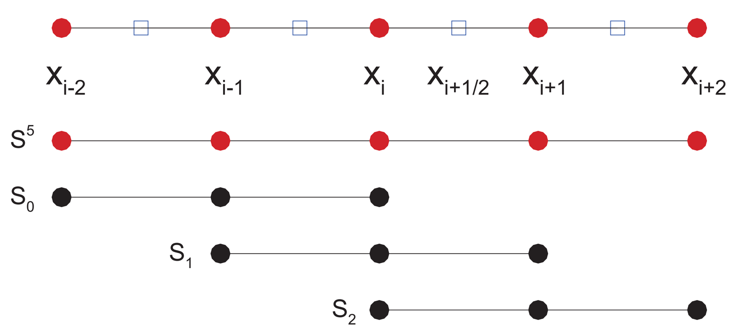
Introduction

For alternative WENO scheme, its polynomial reconstruction procedure is applied to the conservative variables rather than the flux function. Hence, arbitrary monotone fluxes can be used in this frame work and the scheme can enhance the resolution and reduce dissipation. An improved fifth order AWENO-Z scheme has been successfully used for solving nonlinear hyperbolic conservation laws. Here we demonstrate the improved performance of seventh order and ninth order AWENO-Z schemes in terms of accuracy, shock capturing and resolution for complex flow structures.

WENO Interpolation



Given the point values $Q_i = Q(x_i)$ of a function $Q(x)$, we need to find an approximation of $Q(x)$ at the half nodes $x_{i+\frac{1}{2}}$ using the unique polynomial interpolation $p(x)$, in the stencil.
For the sub-stencils, the $(r-1)$ degree interpolation polynomial $Q^{(k)}(x)$ at the cell boundary $x_{i+\frac{1}{2}}$ in substencil S_k can be found as $Q^{(k)}_{i+\frac{1}{2}} = \sum_{j=0}^{r-1} L^{2r-1}_{k+1,j} Q_{i+j+k-(r-1)}, i = 0, ..., N, k = 0, ..., r-1$ with

$$L^{2r-1}_{k+1} = \left[\begin{array}{cccccc} 1 & \frac{1}{2} & (\frac{1}{2})^2 \frac{1}{2!} & \cdots & (\frac{1}{2})^{r-1} \frac{1}{(r-1)!} \end{array}\right] \left[\begin{array}{c} H^k \\ G^k \end{array}\right], \quad (1)$$

$$H^k_m = \begin{cases} 1, & m = r-k \\ 0, & \text{otherwise} \end{cases} \qquad m = 1, ..., r. \quad (2)$$

The expression of G^k is shown in detail later.
For the big stencil S^{2r-1}, the $(2r-1)$th degree polynomial approximation $Q_{i+\frac{1}{2}}$ can be written as $Q_{i+\frac{1}{2}} = \sum_{k=0}^{r-1} d_k Q^{(k)}_{i+\frac{1}{2}}$, where

$$d_k = \frac{1}{2^{2r-2}} \binom{2r-1}{2k}, k = 0, ..., r-1, \binom{n}{m} = \frac{n!}{m!(n-m)!}, \quad (3)$$

which are referred to as the linear weights. Change the linear weights d_k into the nonlinear weights ω_k,

$$\omega_k = \frac{\alpha_k}{\sum_{s=0}^{r-1} \alpha_s}, \quad k = 0, 1, ..., r-1,$$

$$\alpha_k^{JS} = \frac{d_k}{(\beta_k + \epsilon)^p}, \quad \text{or } \alpha_k^{Z} = d_k \left(1 + \left(\frac{\tau_{2r-1}}{\beta_k + \epsilon}\right)^p\right),$$

where β_k are *smoothness indicators* of the stencil S_k, which measures the smoothness of $Q(x)$ in stencil S_k, p is the power parameter, ϵ is the sensitivity parameter to avoid the denominator to be zero, and $\tau_7 = |-\beta_0 - 5\beta_1 + 5\beta_2 + \beta_3|, \tau_9 = |\beta_0 + 4\beta_1 - 10\beta_2 + 4\beta_3 + \beta_4|$.
Usually, $p = 2, \epsilon = 10^{-12}$.
In order to keep the symmetry, β_k can be written as a quadratic bilinear form of polynomial approximations v^k_m to derivatives of $Q(x)$,

$$\beta_k = \langle v^k, C^r v^k \rangle, v^k_m = Q_i^{(m)} \Delta x^m + O(\Delta x^r), m = 1, ..., r-1,$$

$$v^k_m = \sum_{n=1}^{r} G^k_{m,n} Q_{i+k-r+n} = Q_i^{(m)} \Delta x^m + O(\Delta x^r),$$

$$v^k = G^k Q^k, Q^k_n = Q_{i+k-r+n}, n = 1, ..., r.$$

where $\mathbf{C^r}$ can be expressed as

$$\mathbf{C^r_{mn}} = \sum_{q=1}^{r-1} \frac{1}{(m-q)!} \frac{1}{(n-q)!} \frac{1}{p} \left[\left(\frac{1}{2}\right)^p - \left(-\frac{1}{2}\right)^p\right], m, n = q, ..., r-1.$$

where $p = m + n - 2q + 1$ and $\mathbf{C^{r-1}}$ is the leading principal minor of $\mathbf{C^r}$.

WENO Interpolation

$$\mathbf{C^5} = \begin{pmatrix} 1 & 0 & \frac{1}{24} & 0 \\ 0 & \frac{13}{12} & 0 & \frac{7}{160} \\ \frac{1}{24} & 0 & \frac{1043}{960} & 0 \\ 0 & \frac{7}{160} & 0 & \frac{87617}{80640} \end{pmatrix}.$$

G^k is consistent with the classic WENO schemes [1]. Given $\mathbf{G^k}$, one has the identity

$$G^k_{m,n} = (-1)^m G^{r-1-k}_{m,r+1-n}, k = (\lceil \frac{r}{2} \rceil + 1), ..., r-1. \quad (4)$$

The compact formulas for β_k in terms of v^k_m are given for $r = 4, 5$.
$r = 4$,

$$\beta_k = (v^k_1 + \frac{1}{24} v^k_3)^2 + \frac{13}{12} (v^k_2)^2 + \frac{781}{720} (v^k_3)^2.$$

$r = 5$,

$$\beta_k = (v^k_1 + \frac{1}{24} v^k_3)^2 + (v^k_2)^2 + \frac{1}{12}(v^k_2 + \frac{21}{40} v^k_4)^2 + \frac{781}{720} (v^k_3)^2 + \frac{53603}{50400} (v^k_4)^2.$$

We denote $Q^-_{i+\frac{1}{2}}$ and $Q^+_{i+\frac{1}{2}}$ since the stencil S^{2r-1} used to obtain this approximation is biased to the left. The $Q^+_{i+\frac{1}{2}}$ is mirror-symmetric to that for $Q^-_{i+\frac{1}{2}}$, with respect to the target point $x_{i+\frac{1}{2}}$.

Construction of the Scheme

Assuming that $F(Q)$ is a smooth function of Q, we would like to find a consistent numerical flux function $\hat{F}(x)$ such that

$$\frac{1}{\Delta x}(\hat{F}_{i+\frac{1}{2}} - \hat{F}_{i-\frac{1}{2}}) = F(Q(x))_x\big|_{x_i} + O(\Delta x^{2r-1}). \quad (5)$$

Taylor expansion reveals there exists constants a_2, a_4, ... such that

$$\hat{F}_{i+\frac{1}{2}} = F_{i+\frac{1}{2}} + \sum_{k=1}^{r-1} a_{2k} \Delta x^{2k} \left(\frac{\partial^{2k}}{\partial x^{2k}} F\right)_{i+\frac{1}{2}} + O(\Delta x^{2r}), \quad (6)$$

which guarantees $(2r-1)$th order accurate in (5). For example, $r = 5$,

$$a_2 = -\frac{1}{24}, \; a_4 = \frac{7}{5760}, \; a_6 = -\frac{31}{967680}, \; a_8 = \frac{127}{154828800}.$$

The first term of the numerical flux is approximated by $(2r-1)$th order WENO interpolation and these remaining terms by simple central approximation, that is,

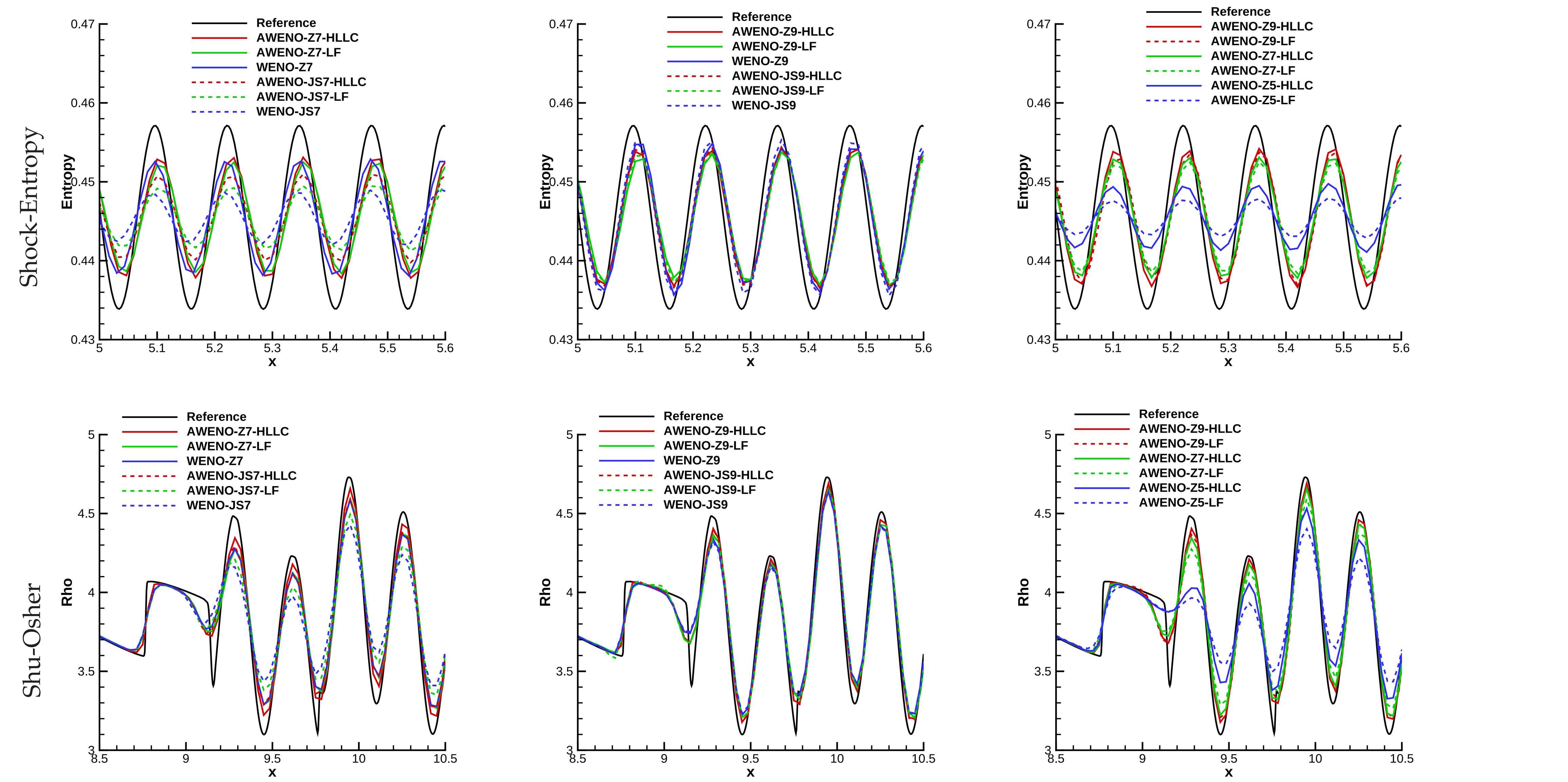
$$\Delta x^{2k} (\frac{\partial^{2k}}{\partial x^{2k}} F)_{i+\frac{1}{2}} = \sum_{k=1}^{2r} b_k F_{i-(r-k)}. \quad (7)$$

Accuracy of Alternative Scheme

Table 1: L^∞ error of Euler equation by seventh and ninth order AWENO schemes with a fixed $\epsilon = 10^{-12}$.

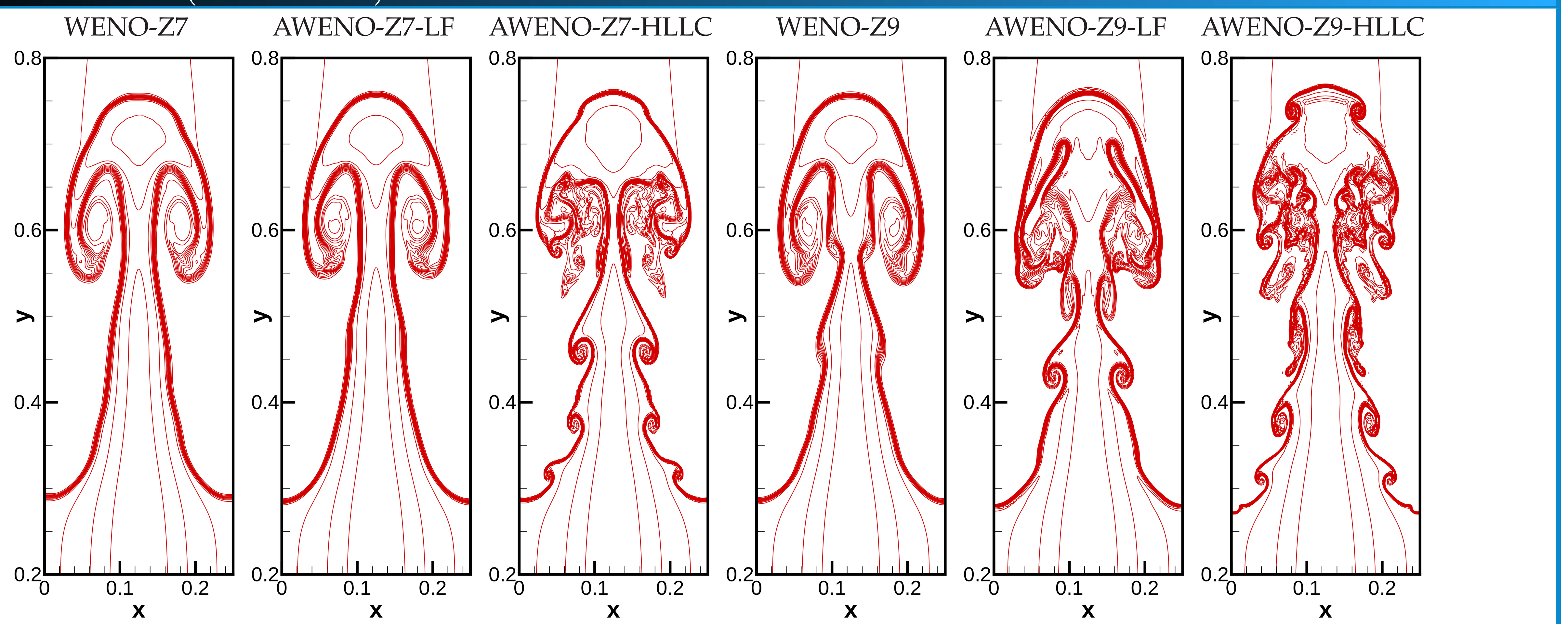
		LF		HLLC	
	N	L^∞ error	Order	L^∞ error	Order
JS7	10	1.3e-3	—	7.9e-4	—
	20	3.0e-5	5.4	1.8e-5	5.5
	40	7.8e-7	5.3	4.3e-7	5.4
Z7	10	1.6e-4	—	7.1e-5	—
	20	1.0e-6	7.3	4.6e-7	7.3
	40	8.1e-9	6.9	3.6e-9	7.0
JS9	10	7.3e-5	—	6.8e-5	—
	20	3.0e-7	7.9	1.4e-7	8.9
	40	6.2e-10	8.9	3.1e-10	8.8
Z9	10	2.8e-5	—	2.4e-5	—
	20	2.1e-8	10.4	9.7e-9	11.0
	40	4.2e-11	9.0	1.9e-11	9.0

1D Problem



Consider the Mach 3 shock-entropy wave interaction problem (N = 1500) and extended Shu-Osher problem (N = 600) to show that that 1) the dissipation error by the AWENO-Z scheme is lower than the AWENO-JS scheme with the same Riemann solver, 2) the resolution of the AWENO-Z scheme with HLLC is higher than with LF solver and higher order AWENO schemes give a much better resolution.

RTI Problem (120×480)



The colored contour of density shows the AWENO scheme with HLLC solver exhibits lesser numerical dissipation and higher order AWENO scheme exhibits higher resolution than lower AWENO scheme for complex flow structures.

Reference

[1] W.S. Don and R. Borges, *Accuracy of the Weighted Essentially Non-Oscillatory Conservative Finite Difference Schemes*, J. Comput. Phys. **250** (2013) 347–372.

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[3] Y. Jiang, C.-W. Shu and M. Zhang, *An alternative formulation of finite difference weighted ENO schemes with Lax-Wendroff time discretization for conservation laws*, SIAM J. Sci. Comput. **35**(2) (2013) A1137–A1160.

Future Work

Organizing paper, studying Ablative Rayleigh-Taylor instability, parallel computing and high accuracy numerical method for conservation laws.

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