High Order Well-Balanced Alternative WENO Scheme Under Gravitational Fields

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Introduction

In this study, a numerical framework of the high order well-balanced finite difference scheme based on the alternative WENO scheme [1] is proposed for the gas dynamics equations under gravitational fields. We employ a special splitting technique for the source term proposed and the reconstruct methods for the conservative variables to maintain the exact C-property, which can be proved theoretically. Here we consider the equations governing the conservation of mass, momentum and energy of an inviscid, non-heat conducting, isotropic fluid.

Notations and Operators

Definition 1 *$\Delta^n[f] = f^{n+1} - f^n$ is the temporal change from step n to step $n+1$.*

Definition 2 *$\delta_1[f] = f^+ + f^-$ and $\delta_2[f] = f^+ - f^-$ are the sum and jump of a function at a given point, respectively.*

Definition 3 *$\mathcal{H} = \frac{1}{2}(\delta_1 - \alpha\delta_2)$ is the Lax-Friedrichs flux operator with the fixed constants α.*

Definition 4 *$\Delta_j[f] = f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}}$ is the divided difference operator.*

Definition 5 *$\Delta_j\mathcal{D}[f] = \mathcal{D}_{j+\frac{1}{2}}[f] - \mathcal{D}_{j-\frac{1}{2}}[f]$, where $\mathcal{D}_{j+\frac{1}{2}}$ is the linear operator of the high order derivative term.*

Definition 6 *$W[\cdot]$ is the nonlinear WENO interpolation operator.*

Definition 7 *Linearized nonlinear WENO operator W^**

Euler Equation

These gas dynamic equations, coupled with a static gravitational potential, given in one space dimension, by

$$Q_t + F(Q)_x = G \tag{1}$$

$$L = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, F(Q) = \begin{bmatrix} \rho u \\ \rho u^2 + P \\ (E+P)u \end{bmatrix}, G = \begin{bmatrix} 0 \\ -\rho\phi_x \\ -\rho u\phi_x \end{bmatrix}.$$

where ρ, u, P are the fluid density, velocity, and pressure, and total energy $E = \frac{1}{2}\rho u^2 + \frac{P}{\gamma-1}$ is the non-gravitational energy which includes the kinetic and internal energy of the fluid. γ is the ratio of specific heats and $\phi = \phi(x)$ is the time independent gravitational potential.

The left and right eigenvectors of the Jacobian of the flux $F(Q)$ are

$$L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(b_2 + \frac{u}{c}) & -\frac{1}{2}(b_1 u + \frac{1}{c}) & \frac{1}{2}b_1 \\ 1 - b_2 & b_1 u & -b_1 \\ \frac{1}{2}(b_2 - \frac{u}{c}) & -\frac{1}{2}(b_1 u - \frac{1}{c}) & \frac{1}{2}b_1 \end{bmatrix},$$

$$R = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ u - c & u & u + c \\ H - uc & \frac{u^2}{2} & H + uc \end{bmatrix},$$

and their corresponding eigenvalues are

$$\Lambda = [\ \lambda_1,\ \lambda_2,\ \lambda_3\ ] = [\ u - c,\ u,\ u + c\ ],$$

where the sound speed c, the enthalpy H,b_1, b_2 are

$$c = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{(\gamma-1)(H - \frac{u^2}{2})},\ H = \frac{E+P}{\rho},\ b_1 = \frac{\gamma-1}{c^2},\ b_2 = \frac{b_1 u^2}{2}.$$

A Steady State of Euler Equation

The special steady state of the form

$$\rho = \exp(-gx), \quad u = 0, \quad P = \exp(-gx),$$

coupled with the linear gravitational potential field $\phi_x = g$. The source term of Eq. can be reformulated as

$$G = AS_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho\exp(gx) & 0 \\ 0 & 0 & \rho u\exp(gx) \end{bmatrix} \begin{bmatrix} 0 \\ \exp(-gx) \\ \exp(-gx) \end{bmatrix}_x,$$

where the gravitational source $-g$ is replaced by $\exp(gx)(\exp(-gx))_x$.

Well-Balanced Condition

Now, we define the set of admissible steady states by

$$\mathbf{G}_E = \{\mathbf{Q} = (\rho, \rho u, E)^T |\ \rho = \exp(-gx),\ u = 0,\ P = \exp(-gx)\}.$$

If $\mathbf{Q} \in \mathbf{G}_E$, then

$$\tilde{\mathbf{L}} = \begin{bmatrix} 0 & -\frac{1}{2c} & \frac{1}{2}b_1 \\ 1 & 0 & -b_1 \\ 0 & \frac{1}{2c} & \frac{1}{2}b_1 \end{bmatrix}, \quad \tilde{\mathbf{R}} = \begin{bmatrix} 1 & 1 & 1 \\ -c & 0 & c \\ H & 0 & H \end{bmatrix}$$

$$\Lambda = [\ -c,\ 0,\ c\ ], \quad \mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- First, $\mathbf{Q}$ and $\mathbf{S}$ used for the WENO interpolation are projected into characteristic fields,

$$\tilde{\mathbf{Q}}_{j+k} = \tilde{\mathbf{L}}\mathbf{Q}_{j+k} = [\ \tilde{q}_1\ \ \tilde{q}_2\ \ \tilde{q}_3\ ]^T_{j+k}, \tag{2}$$

$$\tilde{\mathbf{S}}_{j+k} = \tilde{\mathbf{L}}\mathbf{S}_{j+k} = [\ \tilde{s}_1\ \ \tilde{s}_2\ \ \tilde{s}_3\ ]^T_{j+k}. \tag{3}$$

- Next, using the WENO interpolation procedure, the reconstructed $\tilde{\mathbf{Q}}^\pm_{j+\frac{1}{2}}$ and $\tilde{\mathbf{S}}^\pm_{j+\frac{1}{2}}$ at the cell boundaries become

$$\tilde{\mathbf{Q}}^\pm_{j+\frac{1}{2}} = W^\pm_{j+\frac{1}{2}}[\tilde{\mathbf{Q}}] \triangleq [\tilde{q}_1^\pm, \tilde{q}_2^\pm, \tilde{q}_3^\pm]^T_{j+\frac{1}{2}}, \tag{4}$$

$$\tilde{\mathbf{S}}^\pm_{j+\frac{1}{2}} = W^\pm_{j+\frac{1}{2}}[\tilde{\mathbf{S}}] \triangleq [\tilde{s}_1^\pm, \tilde{s}_2^\pm, \tilde{s}_3^\pm]^T_{j+\frac{1}{2}}, \tag{5}$$

  where $W[\cdot]$ is the nonlinear WENO interpolation operator.

- Then, $\tilde{\mathbf{Q}}^\pm_{j+\frac{1}{2}}$ and $\tilde{\mathbf{S}}^\pm_{j+\frac{1}{2}}$ are projected back into the physical space $\mathbf{Q}^\pm_{j+\frac{1}{2}}$ and $\mathbf{S}^\pm_{j+\frac{1}{2}}$,

$$\mathbf{Q}^\pm_{j+\frac{1}{2}} = \tilde{\mathbf{R}}\tilde{\mathbf{Q}}^\pm_{j+\frac{1}{2}}, \quad \mathbf{S}^\pm_{j+\frac{1}{2}} = \tilde{\mathbf{R}}\tilde{\mathbf{S}}^\pm_{j+\frac{1}{2}}. \tag{6}$$

- According to $\tilde{q}_1 = \tilde{q}_3$, one has

$$\mathbf{Q}^\pm_{j+\frac{1}{2}} = \begin{bmatrix} 2\tilde{q}_1^\pm + \tilde{q}_2^\pm \\ 0 \\ \frac{2\gamma}{\gamma-1}\tilde{q}_1^\pm \end{bmatrix}_{j+\frac{1}{2}}, \quad \mathbf{F}(\mathbf{Q}^\pm_{j+\frac{1}{2}}) = \begin{bmatrix} 0 \\ 2\gamma\tilde{q}_1^\pm \\ 0 \end{bmatrix}_{j+\frac{1}{2}}. \tag{7}$$

A general explicit conservative scheme,

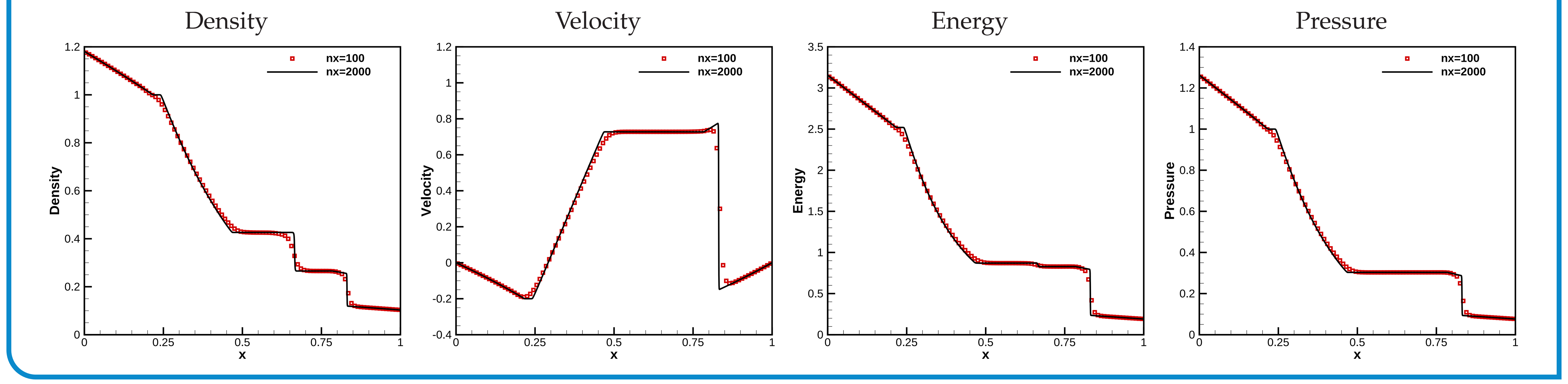
$$\Delta^n[\mathbf{Q}_j] = \lambda(-\Delta_j(\widehat{\mathbf{F}} + \mathcal{D}[\mathbf{F}]) + \mathbf{A}\Delta_j(\widehat{\mathbf{S}} + \mathcal{D}[\mathbf{S}])), \qquad \mathbf{Q}^n_j \in \mathbf{G}_E, \tag{8}$$

$$\widehat{\mathbf{F}} = \frac{1}{2}(\delta_1[\mathbf{F}(\mathbf{Q})] - \alpha\delta_2[\mathbf{Q}]),\ \widehat{\mathbf{S}} = \frac{1}{2}(\delta_1[\mathbf{S}]). \tag{9}$$

Since $\mathbf{Q}^n_j \in \mathbf{G}_E$, $\mathbf{Q}^{n+1}_j \in \mathbf{G}_E$ means $\Delta^n[\mathbf{Q}_j] = 0$. And we have $\Delta_j\mathcal{D}[\mathbf{F}] = \mathbf{A}\Delta_j\mathcal{D}[\mathbf{S}]$ since $\mathcal{D}$ is a linear operator. A well-balanced method is one that balances the flux and source term exactly, i.e. $\Delta_j[\widehat{\mathbf{F}}] - \mathbf{A}\Delta_j[\widehat{\mathbf{S}}] = \mathbf{0}$ for the steady state solution, that is,

$$\Delta_j(\delta_1[\mathbf{F}] - \alpha\delta_2[\mathbf{Q}]) - \mathbf{A}\Delta_j(\delta_1[\mathbf{S}]) = \mathbf{0}. \tag{10}$$

1D Sod Problem under Gravitational Field at Time $t = 0.2$ [2]



First Condition

The first element of $\mathbf{F}(\mathbf{Q}^\pm_{j+\frac{1}{2}})$ and right term in Eq. (10) are zero, then,

$$\Delta_j(\delta_2[2\tilde{q}_1 + \tilde{q}_2]) = 0. \tag{11}$$

To satisfy the condition (11), we need to modify the LF solver (9) as

$$\widehat{\mathbf{F}} = \frac{1}{2}(\delta_1[\mathbf{F}(\mathbf{Q})] - \bar{\alpha}\delta_2[\bar{\mathbf{Q}}]). \tag{12}$$

The variable $\mathbf{Q}^\pm$ and α can be modified by $\bar{\mathbf{Q}}^\pm = \tilde{\mathbf{R}}W[\tilde{\mathbf{L}}(\mathbf{Q}\exp(gx))]$ and $\bar{\alpha} = \alpha\max\exp(-gx)$. With the steady state Eq. (), it follows

$$\mathbf{Q}\exp(gx) = \begin{bmatrix} \exp(-gx)\exp(gx) \\ 0 \\ \frac{1}{\gamma-1}\exp(-gx)\exp(gx) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{\gamma-1} \end{bmatrix}, \quad \bar{\mathbf{Q}} = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{\gamma-1} \end{bmatrix},$$

then $\Delta_j(\delta_2[\bar{\mathbf{Q}}]) = 0$ and $\Delta_j(\delta_2[2\bar{q}_1 + \bar{q}_2]) = 0$.

Second Condition

The second condition becomes

$$\Delta_j(\delta_1[2\gamma\tilde{q}_1 - \tilde{s}_2]) = 0,$$

$$2\gamma W_F^{1,3}[\rho\frac{1}{2\gamma}] - \sqrt{\gamma}\left(W_S^3[\rho\frac{\gamma - 1 + \sqrt{\gamma}}{2\gamma}] - W_S^1[\rho\frac{\gamma - 1 - \sqrt{\gamma}}{2\gamma}]\right) = C.$$

Using the linearized nonlinear WENO operator by ρ, W^ρ, we have

$$2\gamma W_F^{1,3}[\rho\frac{1}{2\gamma}] - \sqrt{\gamma}\left(W_S^3[\rho\frac{\gamma - 1 + \sqrt{\gamma}}{2\gamma}] - W_S^1[\rho\frac{\gamma - 1 - \sqrt{\gamma}}{2\gamma}]\right) = 0.$$

Third Condition

By the modification in Eq. (12), we have $\Delta_j(\delta_2[\bar{\mathbf{Q}}]) = 0$. Thus, Eq. (12) gives a well-balanced solution in the third equation.

Reference

[1] B.-S. Wang, P. Li, Z. Gao and W.S. Don, *An improved fifth order alternative WENO-Z finite difference scheme for hyperbolic conservation laws*, J. Comput. Phys. **374**, 469–477 (2018).

[2] Y.L. Xing and C.-W. Shu, *High order well-balanced WENO scheme for gas dynamics equations under gravitational fields*, J. Sci. Comput. **54** (2012) 645–662.

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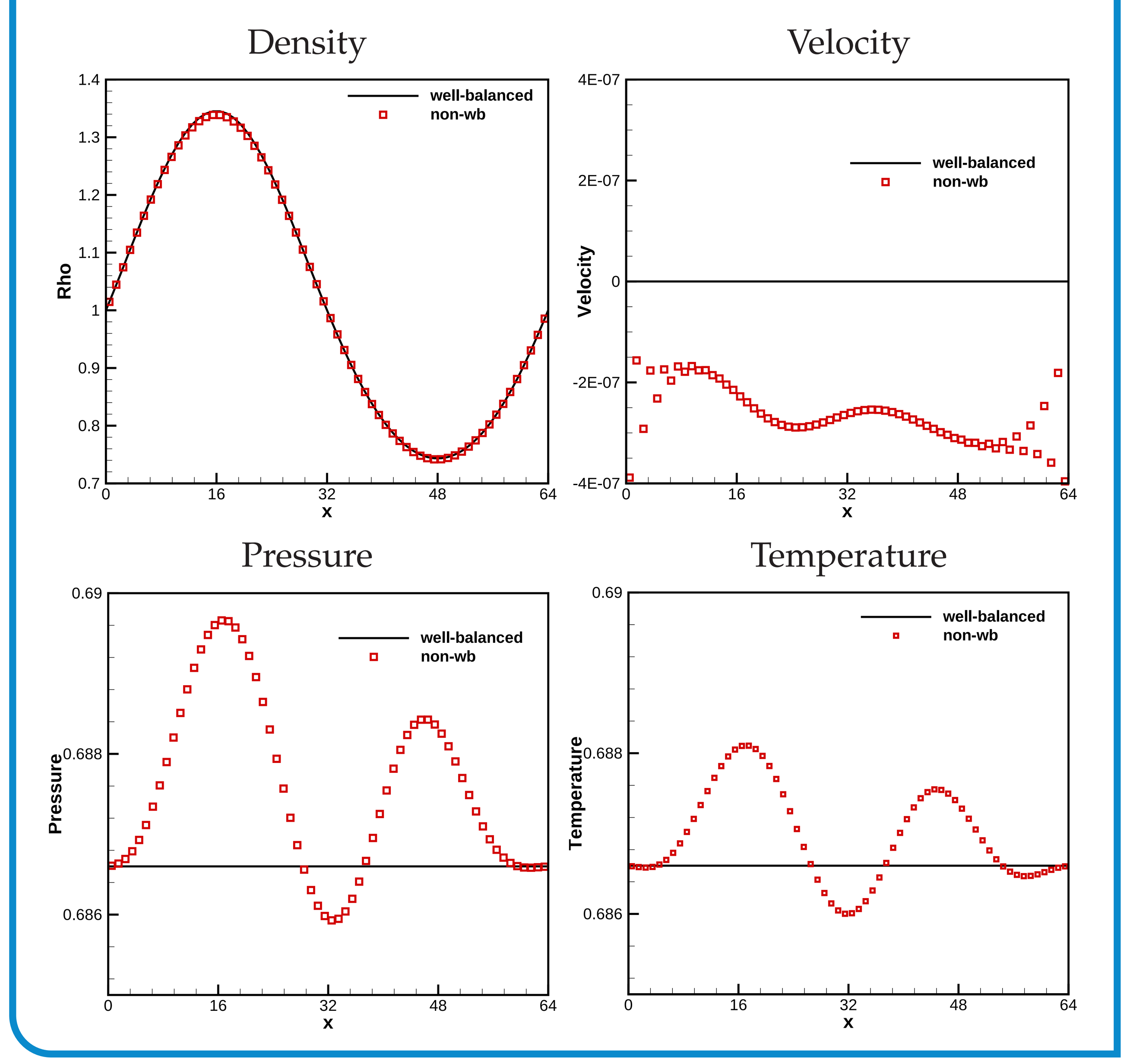
1D Fixed External Potential [2]



2D RayleighĺCTaylor Instability