A Spectral Study on the Dissipation and Dispersion of the WENO Schemes

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Feilin Jia, Zhen Gao, Wai Sun Don

Introduction

We investigate the spectral properties (dispersion and dissipation) of the fifth order nonlinear schemes (the WENO-JS and WENO-Z schemes) using both the approximate dispersion relation (ADR) and nonlinear spectral analysis (NSA). The presence of the sensitivity parameter ϵ and the influences it has on the nonlinearity of the WENO schemes are taken into consideration of the analysis. We propose an ϵ-adaptive WENO-Z scheme, adapting the value of the sensitivity parameter $\epsilon(x,t)$ spatially and temporally based on the local smoothness of the solution. This adaptive scheme holds a promise that one can obtain the desirable spectral properties of an optimal order linear scheme while capturing discontinuities essentially non-oscillatory by the WENO schemes.

WENO schemes

The nonlinear weights ω_k for the fifth order WENO schemes [1, 2] are

$$\omega_k = \frac{\alpha_k}{\sum_{l=0}^{2}\alpha_l}, \quad (1)$$

$$\alpha_k = \begin{cases} d_k\left(\frac{1}{\beta_k+\epsilon}\right)^p & \text{(WENO-JS)} \\ d_k\left(1+\left(\frac{\tau_5}{\beta_k+\epsilon}\right)^p\right) & \text{(WENO-Z)} \end{cases}, \quad (2)$$

where d_k are the ideal weights which yield the optimal order upwinded central scheme (UW) for smooth function. β_k are the lower order local smoothness indicator and the optimal order global smoothness indicator $\tau_5 = |\beta_0 - \beta_2|$ measures the higher derivatives of function. Taylor series expansion shows, if $f_i' \neq 0$, that β_k is of $O(\Delta x^2)$ and τ_5 is of $O(\Delta x^5)$. $p(=2)$ is the power parameter.

The sensitivity parameter ϵ $(10^{-6} - 10^{-16})$ influences the nonlinear adaptation of the WENO schemes and their spectral properties.

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Approximate Dispersion Relation (ADR)

The performance of the ADR and comparison between solutions computed by different schemes in solving a linear advection problem with an initial condition $Q(x,0) = \sin(x)$ in a periodical domain are shown in Fig. 1. We observe that, due to the nonlinear stencils adaptation effect, the spectral properties of the WENO schemes are different from those of the linear upwinded central finite difference scheme (UW). The WENO-Z scheme is less dispersive and dissipative than the WENO-JS scheme at all wavenumbers.

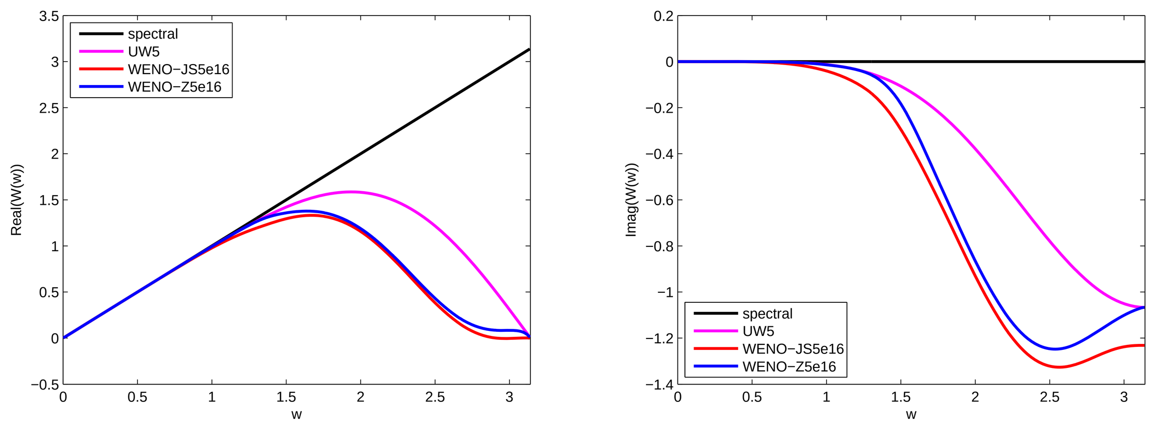

Figure 1: The spectral properties by the ADR.

Here the initial condition contains a broad range of wavenumbers. As shown in Fig. 2, the amplitude spectrum computed by the WENO-JS scheme shows it is more dissipative than that by the WENO-Z scheme, which is almost identical to that computed by the UW scheme in the low-medium range of wavenumbers. The spurious high wavenumbers generated by WENO-JS scheme is larger by a factor of ten than those by WENO-Z scheme.

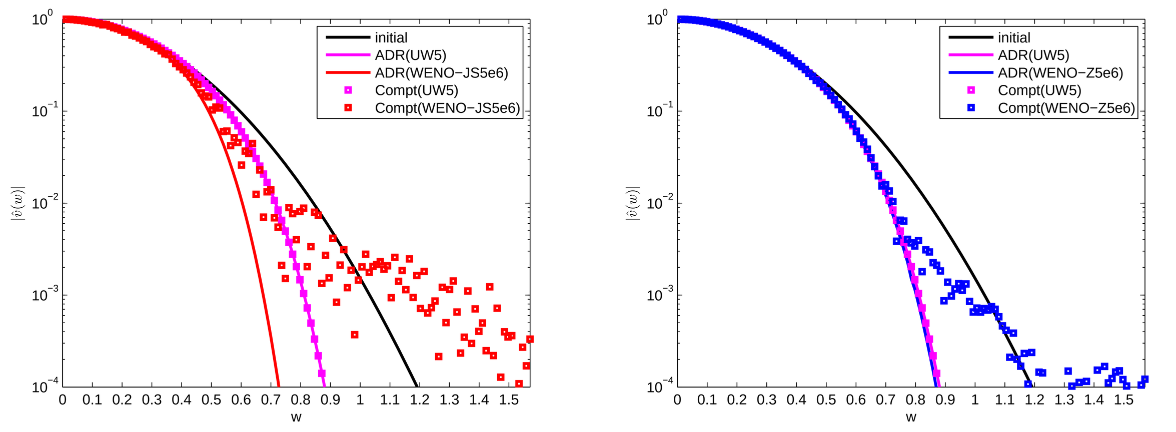

Figure 2: Amplitude spectra predicted by the ADR and computed by the UW scheme and the WENO schemes with $\epsilon = 10^{-6}$.

Nonlinear Spectral Analysis (NSA)

In Fig. 3, the statistics of the spectral properties of the WENO schemes are shown. The initial broadband wave is defined as

$$\hat{u}(\omega) = \pm\sqrt{U_0}\,\omega^{\frac{\alpha}{2}}\frac{1+i\tan(\theta(\omega))}{\sqrt{1+\tan^2(\theta(\omega))}}, \qquad |\omega| \le \pi.$$

where α is $-5/3$, $\theta(\omega)$ is the random phase.
Although the spectral properties of the WENO schemes behave similarly, the WENO-Z scheme is always less dissipative for all wavenumbers.

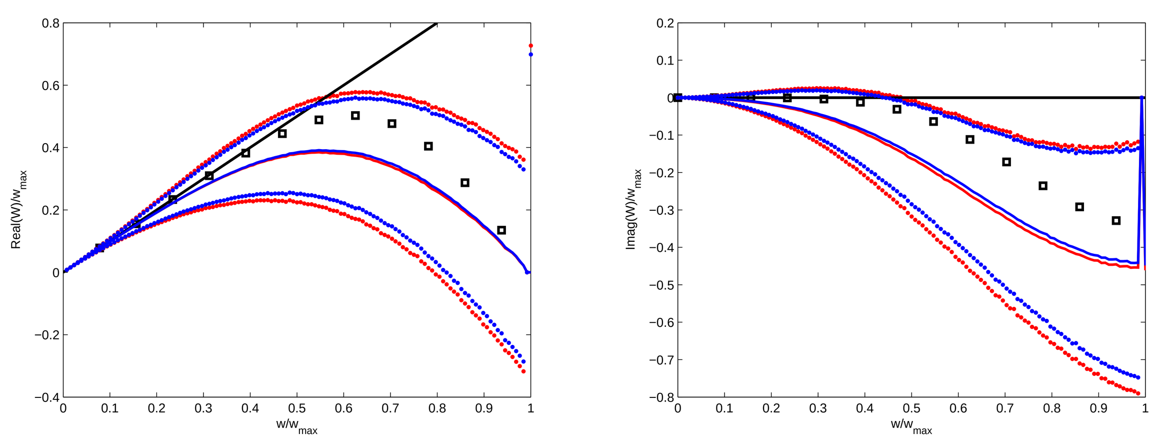

Figure 3: The spectral properties by the NSA with $\epsilon = 10^{-16}$.

In Fig. 4, the error of energy $\Delta E(\omega,t)$ and the energy of error $E_\Delta(\omega,t)$ defined as

$$\Delta E(w,t) = \frac{\Delta(|\hat{u}|^2)}{\gamma}, \quad E_\Delta(w,t) = \frac{|\Delta(\hat{u})|^2}{\gamma},$$

in a long-time advection of a broadband wave as computed by the WENO schemes are shown. γ is the normalization factor. The WENO-Z scheme is less dissipative and generates less spurious high wavenumbers.

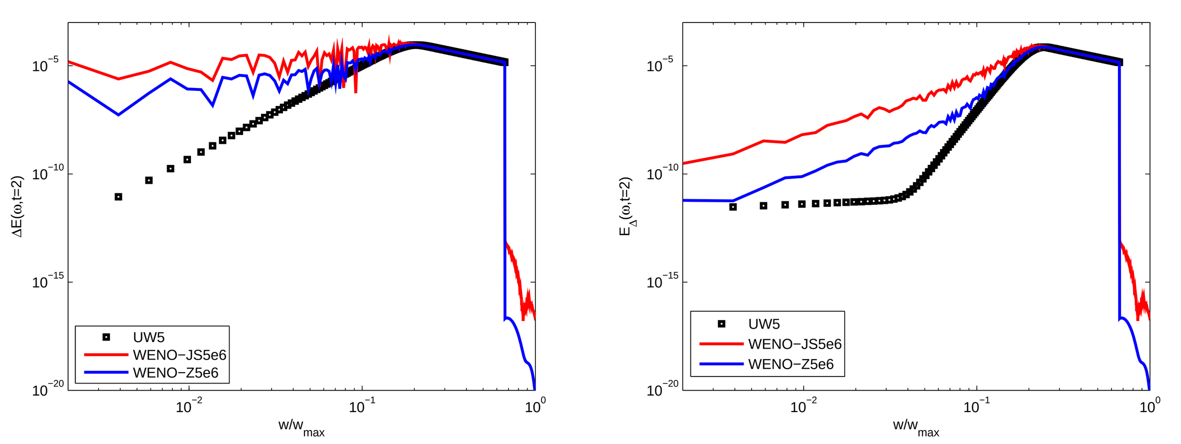

Figure 4: The $\Delta E(\omega,2)$ and $E_\Delta(\omega,2)$ of the UW scheme and the WENO schemes with $\epsilon = 10^{-6}$.

$\epsilon-$adaptive WENO-Z scheme

We propose an ϵ-adaptivity technique, that is

$$\epsilon(x,t) = \begin{cases} \epsilon_{max}, & \text{if } \tau_5 \le S_{min} \\ \exp(k_1\ln(\tau_5)+k_2), & \text{if } S_{min} \le \tau_5 \le S_{max}, \\ \epsilon_{min}, & \text{if } \tau_5 \ge S_{max} \end{cases}$$

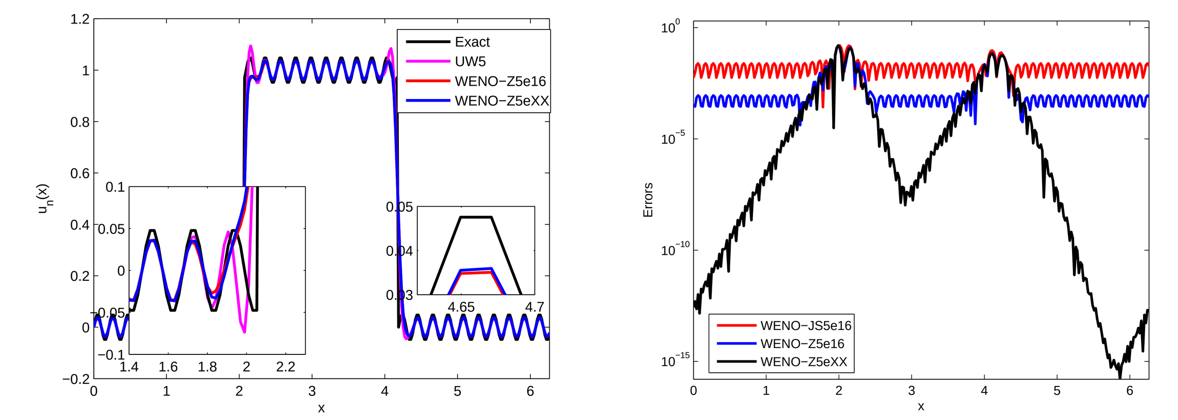

Figure 5: Linear advection of a square wave by the UW scheme and the WENO schemes.

From Fig. 5, the ϵ-adaptive WENO-Z scheme agrees well with that by the WENO-Z scheme around discontinuities and that by the UW scheme in smooth region.

Conclusion

The advantages of the WENO-Z scheme over the WENO-JS scheme are:

1. less dispersive, less dissipative and less sensitive to random phases,
2. preserves energy spectrum better, and
3. generates less spurious high wavenumbers.

Publication

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