High Resolution Schemes and Well Balanced Method of Euler Equations

The Third Summer Workshop in Advanced Research in Applied Mathematics and Scientific Computing 2015, OUC, China

Ti'en-Hsiang Kao, Zhen Gao, Wai-Sun Don

Introduction

It goes without saying that the hight resolution numerical schemes are booming during recently decades unprecedentedly because of both analytical as well as practical demands. As far as people are concerned, an extraordinary well-know spatial finite difference scheme of these is so-called Weighted Essentially Non-oscillatory scheme, or WENO scheme. WENO scheme is based on traditional Essentially Non-oscillatory(ENO) scheme, which is specialized for the discontinuities such as shock waves. Apart from the property of capturing discontinuous region, WENO also make full use of information to maintain high order accuracy.

WENO was firstly introduced in the middle of 1990's with two highly intuitive paper by Xu-Dong Liu et al.(1994) and Jiang, G. S. and Shu, C. W.(1995). The former is considered to be the significant breakthrough for it introduced a far-reaching idea, while the latter refines it(mainly on smoothness measurement) so that obtains a high resolution without waste any information. In 2008, Rafael Borges et al. developed WENO-Z scheme with less dissipation and higher resolution than the classical WENO by using a more reasonable smoothness indicator. Recently, Rafael Borges came up with a new WENO-Z plus method that it constructed a new smoothness indicator with an empirical parameter. Thus more physics dissipation occurs at high frequency region.

Euler system is one of the most intriguing issue in the field of numerical solution of partial differentiate equations for its prevailing applications and conservative structure. In the workshop, we concentrate mainly on one-dimensional Euler system in consideration of time, and can reach the essence of the basic ideas of scientific computing. For both specific and universality, the source term is introduced by gravitational field.

To keep the advantage of high resolution, well-balanced method is used. In short, the aim is to apply same WENO structure on both spatial derivative term and source term, which means the non-linear weight of source term should be the same as the spatial derivative term. The classic Lax-Friedrich global flux-splitting(LF) is performed in the workshop.

Methodology

WENO scheme: WENO scheme derived from ENO, using a convex combination of the candidate stencils instead of only pick the most ideal one, which is an ENO method. This high resolution scheme to performs best when dealing with the hyperbolic conservation laws with the property of capability of continuities as well as high resolution(order of 2r-1, r is the number of stencils). Considering its popularity and the limited space we would not like to discuss this method in detail.

Well-balanced method: When it comes to hyperbolic equations such as Euler equations with source terms, it is a challenge to avoid the error induced by the numerical schemes. Well-balanced method is a tricky approach to use the same WENO scheme for both source terms and spatial derivatives.

The one-dimensional Euler equations are

$$\begin{aligned}&\rho_t + (\rho u)_x = 0\\&(\rho u)_t + (\rho u^2 + p)_x = -\rho\phi_x\\&E_t + ((E+p)u)_x = -\rho u\phi_x\end{aligned} \qquad (1)$$

The simplest and most commonly encountered case is the linear gravitational potential field: $\phi_x = g$. The first step is to rewrite the source term

$$\begin{aligned}&\rho_t + (\rho u)_x = 0\\&(\rho u)_t + (\rho u^2 + p)_x = -\rho\, exp(gx)(exp(-gx))_x\\&E_t + ((E+p)u)_x = -\rho u\, exp(gx)(exp(-gx))_x\end{aligned} \qquad (2)$$

where the gravitational source $-\rho g$ is replaced by $\rho exp(gx)(exp(-gx))_x$, and $-\rho u g$ is treated in the same way. Thus, the source term consists of the gradient form(spatial derivative), which corresponds to numerical flux term. Then, we use WENO-Z to discretize the numerical flux term, and apply the non-linear weights to the source term exactly on those candidate stencils. Now that numerical flux and the source using the same numerical scheme, and the error induced by it cancels naturally, maintaining the high resolution property.

Numerical Results

The test case is the standard Sod test, coupling with the gravitational field. The computational domain is set as [0,1], and the initial conditions are given by

$$\begin{aligned}&\rho = 1 \quad\quad , u = 0, p = 1, \quad if \quad x \le 0.5\\&\rho = 0.125\,, u = 0, p = 0.1, \; if \quad x \ge 0.5\end{aligned} \qquad (3)$$

The gravitational field ϕ takes a value of $g = \phi_x = 1$, and $\gamma = 1.4$. And Time discretization is by the third order TVD Runge-Kutta time discretization, the CFL number is set to be 0.5. We compute this problem using our well-balanced finite difference WENO method with reflection boundary conditions and 100 uniform meshes. The solutions at time $t = 0.2$ with a reference solutiopn computed by traditional finite difference WENO method with much refined 2000 uniform meshes to provide a comparison.

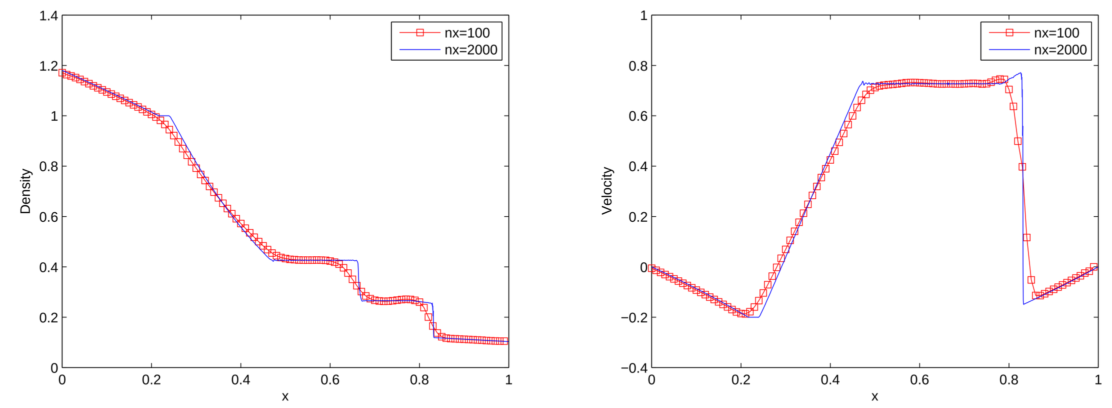

Figure 1: Left: density distribution .Right: velocity distribution.

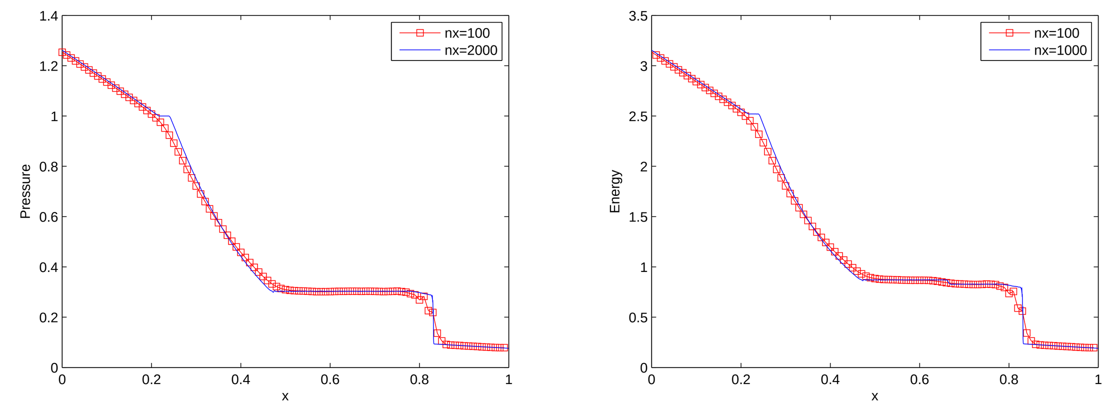

Figure 2: Left: pressure distribution .Right: energy distribution.

From all appearance our solution of numerical experiment is close to the reference solution, showing two salient shock waves. We can conclude from figures above that well-balanced method with finite difference WENO-Z scheme is capable of describe solutions more physically, and can save numerous calculation cost.

Future work

First of all, extend our method to multi-dimensional case with any given gravitational field. In addition to that, characteristic decomposition will be used.

Then, experiment with several flux-splitting technique and access their advantages together with disadvantages.

Last but not least, try to apply the work above on physical oceanography(e.g. shallow water equations and Navier-Stokes equations) or astrophysics(plasma).

Acknowledgement

I would like to appreciate Prof. Wai-Sun Don and Prof. Gao Zhen for their instructions as well as enlightenments, especially for a non-mathematics undergraduate like me. Moreover, I must express my gratitude to Mr. Li Peng, senior Wen Xiao and Zhu Qiangqiang, from whom I learned a lot. Last but not least, I cherish these two months time with everyone of the summer workshop from deep my heart. Auld Lang Syne!

Funding

- School of Mathematical Sciences, OUC.
- Startup grant by Ocean University of China (Don).
- Natural Science Foundation of Shandong Province (ZR2012AQ003).
- National Natural Science Foundation of China (11201441).

Reference

1. Jiang, G. S., & Shu, C. W. (1995). *EEfficient implementation of weighted ENO schemes*(No. ICASE-95-73). INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING HAMPTON VA.
2. Borges, R., Carmona, M., Costa, B., & Don, W. S. (2008). *An improved weighted essentially non-oscillatory scheme for hyperbolic conservation laws.* Journal of Computational Physics, 227(6), 3191-3211.
3. Xing, Y., & Shu, C. W. (2013). *High order well-balanced WENO scheme for the gas dynamics equations under gravitational fields.*Journal of Scientific Computing, 54(2-3), 645-662.