Non-intrusive ROM of convection dominated flows using ANN

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Introduction

In this work, we regard the governing equations of RTI as a parameterized time-dependent PDE with parameterized initial conditions. The amplitude of the initial perturbation waves (A) and time (t) are considered as the parameters in a two-dimensional parameter space. An adaptive sampling (AS) method in time is proposed to reduce the number of samples in the parameter space during the linear regime of the RTI. In this light, the reduced order modeling (ROM) methods have been taken into consideration due to its effectiveness in solving the parameterized problems.

Featuring an offline-online operational framework, the reduced basis method (RBM) is a powerful technique for the ROM methods of the parameterized problems. In general, RBM aims to approximate any member of the solution manifold with a low number of reduced basis (RB) functions.

To construct the ROM, which is a linear combination of the RB functions, the POD is applied to generate the RB functions from the snapshots. The corresponding coefficients of the RB functions are then computed by a training with the ANN. Once the ROM is built, the desired solution with a given parameter can be recovered online efficiently with a slight loss of accuracy.

Reduced Basis Method (RBM)

The general one-dimensional formulation of the well-posed parameterized time-dependent problem is given by

$$\mathcal{L}[Q(x,t;\nu)]+\mathcal{N}[Q(x,t;\nu)]=S(x,t;\nu), \qquad (1)$$

$$(x,t,\nu)\in\Omega\times\mathcal{T}\times\mathcal{W},$$

with some properly defined initial and boundary conditions. $\mathcal{L}[\cdot;\mu]$ and $\mathcal{N}[\cdot;\mu]$ are the linear and nonlinear operators with respect to x and t, respectively. RBM seeks an approximate solution to problem (1) as a linear combination of parameter independent reduced basis (RB) functions $\{\psi_1,\cdots,\psi_L\}$, i.e.,

$$Q(x;\mu)\approx Q_{rb}(x;\mu)=\sum_{i=1}^{L}c_i(\mu)\psi_i(x)\in V_{rb}, \qquad (2)$$

where $\vec{a}(\mu)=[a_1(\mu),\cdots,a_L(\mu)]^T\in\mathbb{R}^L$ is the vector of reduced coefficients, $V_{rb}=\text{span}\{\psi_1,\cdots,\psi_L\}$ is the reduced space. Now, the goal is to find the RB ψ_i and the corresponding coefficients c_i, and to recover the reduced-order solution efficiently.

POD

The proper orthogonal decomposition (POD) is one of the methods to generate the RB. Consider the snapshots matrix $\mathbf{Q}\in\mathbb{R}^{M\times N}$ gathering the nodal values of the snapshots in a column-wise sense, i.e.,

$$\mathbf{Q}=[\vec{Q}_h(\mu_1)|\cdots|\vec{Q}_h(\mu_N)], \qquad (3)$$

where the snapshots are the full-order solutions of Eq. (1) with parameters $\{\mu_1,\cdots,\mu_N\}$.
Then each vector from $\mathbf{Q}$ can be written as

$$\vec{Q}_h(\mu_i)\approx\mathbf{\Phi}\vec{c}_i, \qquad i=1,\cdots,N, \qquad (4)$$

where $\mathbf{\Phi}=[\psi_1(x),\cdots,\psi_L(x)]\in\mathbb{R}^{M\times L}$, $\vec{c}_i=\vec{c}(\mu_i)$.

Coefficients: Minimizing the error

$$e=\sum_{i=1}^{N}\left\|\vec{Q}_h(\mu_i)-\sum_{j=1}^{L}c_{ij}\vec{\psi}_j(x)\right\|^2,$$

one can obtain

$$c_j(\mu_i)=\vec{Q}_h(\mu_i)^T\vec{\psi}_j(x). \qquad (5)$$

As for the **RB**, one can construct the correlation matrix

$$\mathbf{D}=\mathbf{Q}^T\mathbf{Q}. \qquad (6)$$

Then the RB take the form

$$\psi_i=\mathbf{Q}\vec{v}_i\lambda_i^{-1/2}, \qquad (7)$$

where $\vec{v}_i$ is the i^{th} eigenvector of $\mathbf{D}$ with the corresponding eigenvalues λ_i taken in a decreasing order. Therefore, the RB matrix can be written as

$$\mathbf{\Phi}=[\psi_1,\cdots,\psi_L]. \qquad (8)$$

RBM using ANN

We applied the MLP neural network to obtain the mapping relationship between the parameters μ and the coefficients $\vec{c}(\mu)=\mathbf{\Phi}^T\vec{Q}_h(\mu)$, that is,

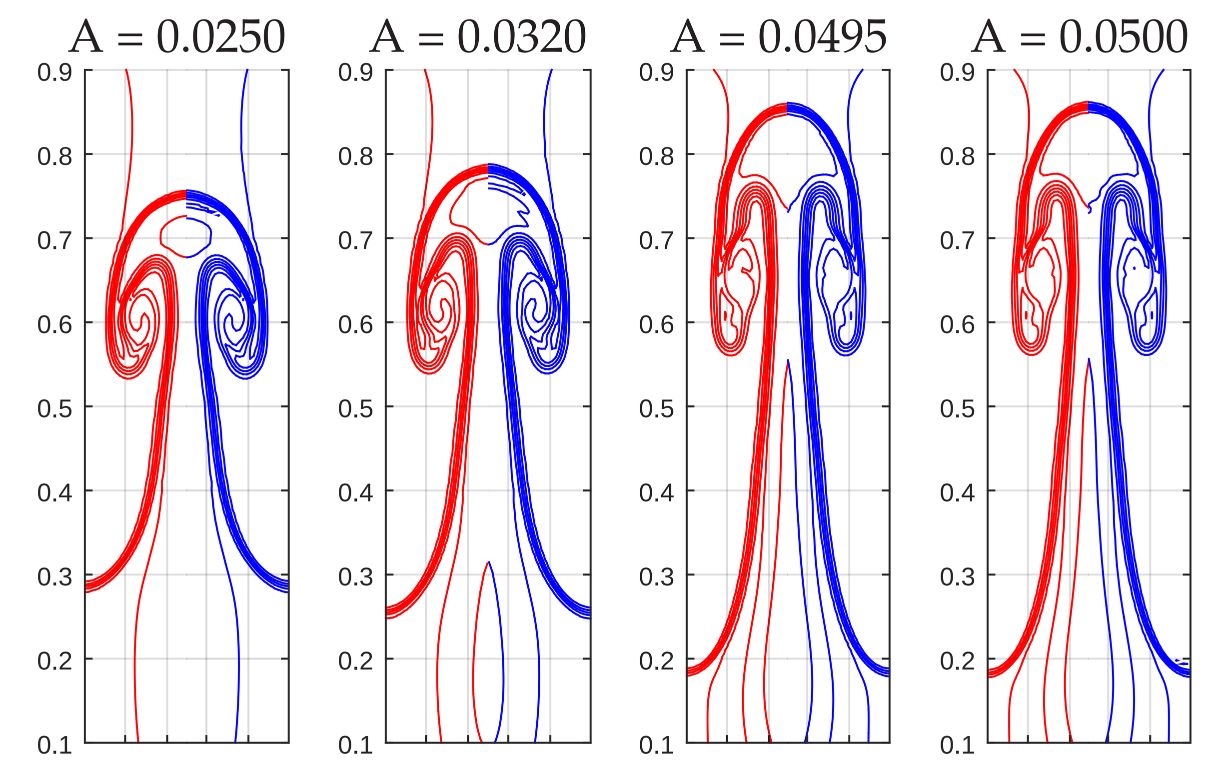
- The training input: $\mu\in\mathcal{P}$.
- The training output: $\mathbf{\Phi}^T\vec{Q}_h(\mu)$.

Then, given a new parameter μ^*, the associated reduced-order solution is simply given by

$$\vec{Q}_{rb}(\mu^*)=\mathbf{\Phi}\mathbf{\Phi}^T\vec{Q}_h(\mu^*). \qquad (9)$$

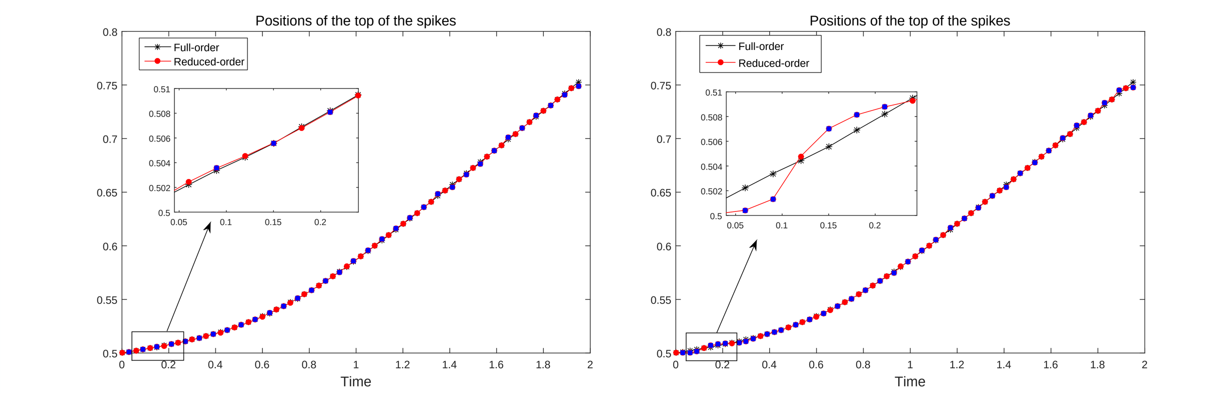
Here, the **cascade-forward** network is used. It includes a weight connection from the input to each layer and from each layer to the successive layers, which might improve the speed of the network learning the desired relationship.

RTI- Amplitude A

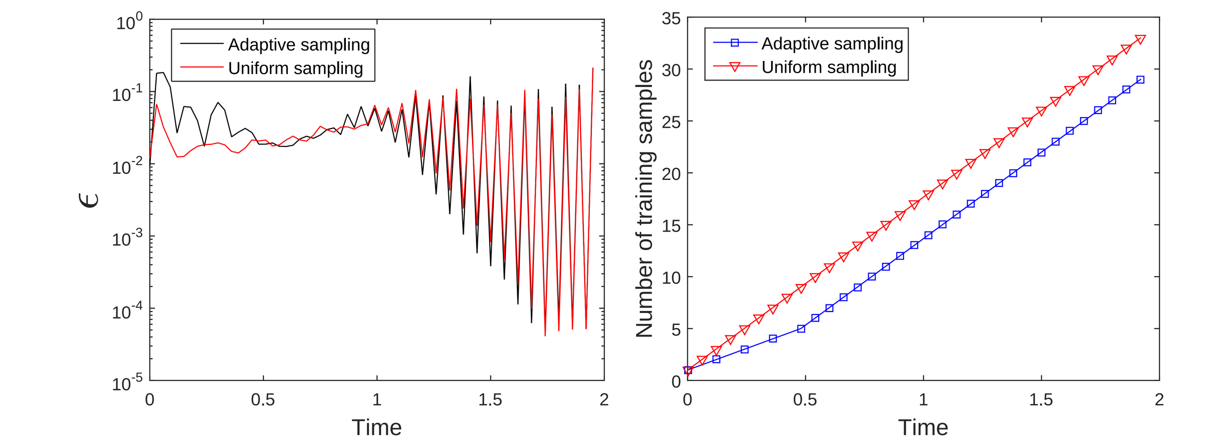


red: full-order solutions, blue:reduced-order solutions.

RTI- Time t



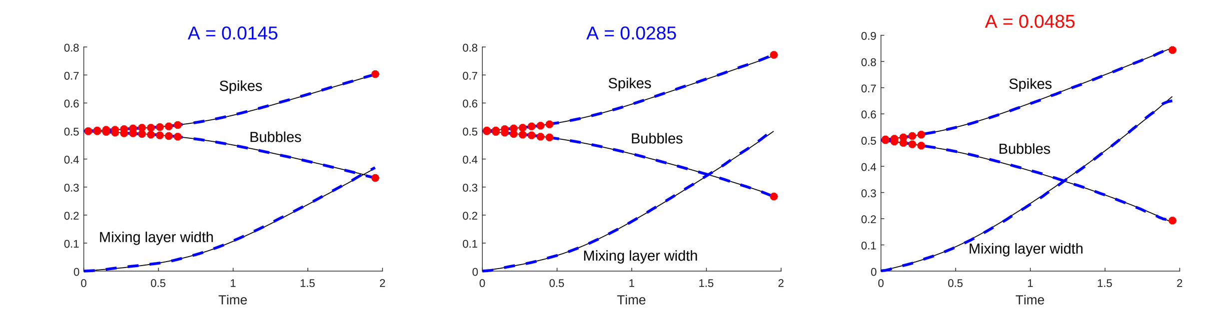
Left: Uniform sampling, Right: Adaptive sampling.



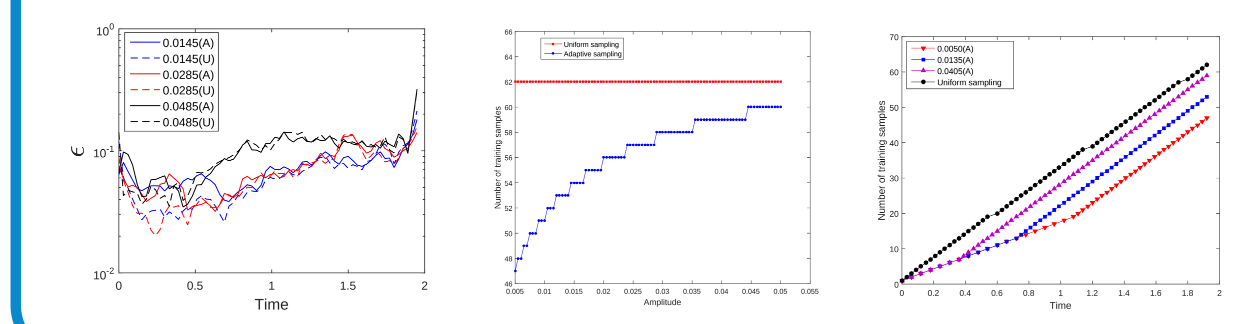
Relative error

$$\epsilon=\frac{\|\vec{Q}_h(\mu)-\vec{Q}_{rb}(\mu)\|_\infty}{\|\vec{Q}_h(\mu)\|_\infty}.$$

RTI- (A,t)



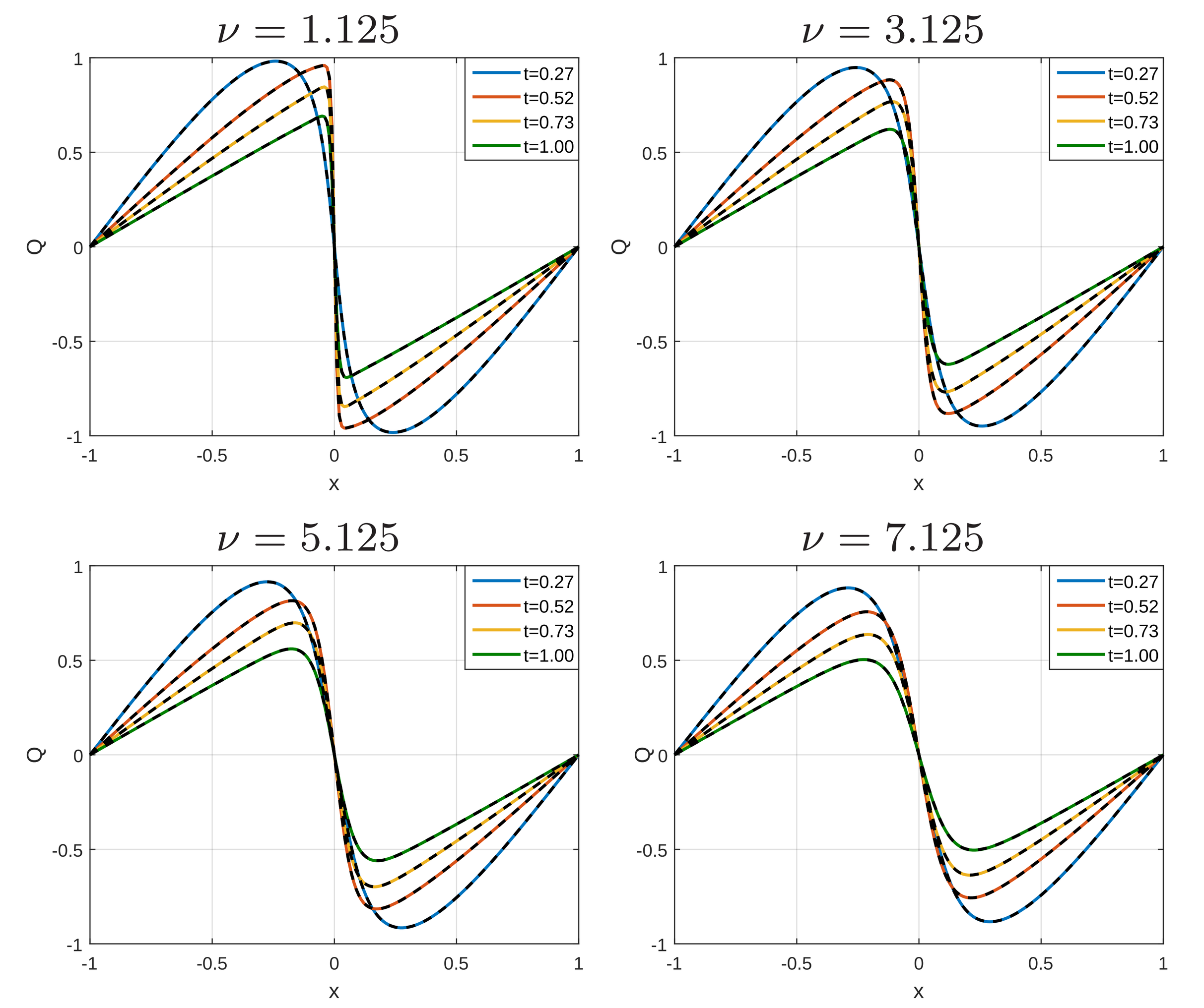
Solid: full-order solutions, Dashed: adaptive solutions.



1-D viscous Burgers Equation

$$Q_t+\left(\frac{Q^2}{2}\right)_x=\frac{\nu}{50\pi}Q_{xx},\ (x,t,\nu)\in(-1,1)\times(0,1]\times[1,7.5],$$

$$Q(-1,t;\nu)=Q(1,t;\nu)=0, \qquad Q(x,0;\nu)=-\sin(\pi x),$$



Solid: full-order solutions, Dashed: reduced-order solutions.

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