Scale invariant WENO scheme with modified Z-type nonlinear weights for solving hyperbolic conservation law

Wai Sun Don, Run Li, Bao-Shan Wang, Ying-Hua Wang

Abstract

A simple and effective modification of the Z-type nonlinear weights in a high order weighted essentially non-oscillatory (WENO) scheme that achieves both the optimal order of accuracy at high order critical points with a smooth function (Cp-property) and the scale invariant property has been designed. A scale invariant WENO scheme dictates that the adaptive WENO reconstruction/interpolation procedure should be valid and independent of the scaling of the data and the sensitivity parameter (Si-property). It is particularly necessary when the scaling is very small in order to avoid violating the essentially non-oscillatory (ENO) property at a discontinuity. In the modified Z-type nonlinear weights of the WENO-D scheme [Wang et al., J. Sci. Comput. 81 (2019) 1329–1358], the scaling dependency of both the modifier function and the sensitivity parameter is removed by a descaling function, which is a global average of the function values. It renders the modified WENO-D (WENO-Dm) scheme satisfies both the Cp-property and the Si-property simultaneously.

Properties

Definition 1 Critical Point: *If $f'(x_c) = \ldots = f^{n_{cp}}(x_c) = 0$ but $f^{n_{cp}+1}(x_c) \neq 0$, the smooth function $f(x)$ is said to have a critical point of order $n_{cp} = n$ at x_c.*

Definition 2 Cp_ε-Property: *For any given power parameter p and a variable sensitivity parameter $\varepsilon = \varepsilon(\Delta x)$, which is a function of grid spacing Δx, a $(2r-1)$ order WENO scheme is said to be satisfying the Cp_ε-property if the WENO scheme achieves its optimal order of accuracy in approximating the first derivative of a smooth function regardless of critical points up to the r order.*

Definition 3 Cp-Property: *For any given power parameter p and sensitivity parameter ε, a $(2r-1)$ order WENO scheme is said to be satisfying the Cp-property if the WENO scheme achieves its optimal order of accuracy in approximating the first derivative of a smooth function regardless of critical points up to the r order.*

Definition 4 Si-Property (Scale Invariant/Self Similarity): *The function $f(x)$ is said to be scale invariant if $f(\kappa x) = \kappa\, f(x)$, for all scaling factor κ.*

WENO-Dm Reconstruction

The $(2r-1)$ degree polynomial approximation $q_{i\pm\frac{1}{2}}$ is built through a convex combination of the interpolated values $q^k(x_{i\pm\frac{1}{2}})$ at $x_{i\pm\frac{1}{2}}$ on the substencil $S_k = \{x_{i-(r-1)+k}, \cdots, x_{i+k}\}$, that is, $q_{i\pm\frac{1}{2}} = \sum_{k=0}^{r-1}\omega_k q(x_{i\pm\frac{1}{2}})$, where the nonlinear weights ω_k are given as

$$\omega_k = \frac{\alpha_k}{\sum_{j=0}^{r-1}\alpha_j}, \quad \alpha_k = d_k\left(1 + \Phi\left(\frac{\tau_{2r-1}}{\beta_k + \varepsilon\eta^2}\right)^p\right), \qquad (1)$$

WENO-Dm Reconstruction

where the linear weights $\{d_0 = \frac{1}{10}, d_1 = \frac{3}{5}, d_2 = \frac{3}{10}\}$ and power parameter $p = 2$ are used. The modifier function Φ is redefined as

$$\Phi = \min\{1, \phi/\mu\}, \phi = \sqrt{|\beta_0 - 2\beta_1 + \beta_2|}, \mu = \|\mu\| + 10^{-40}.$$

The Si-property requires that the descaling function μ must have the same scaling (e.g. dimensional units) of $\sqrt{\beta_k}$ in order for the terms ϕ/μ and $\varepsilon\mu^2$ to be scale invariant. This will lead to a scale invariant form of nonlinear weight α_k. Hence, we define the descaling function μ to be the global average of absolute values of the function values $\{f_i, i = 0, \ldots, N\}$ being reconstructed, that is,

$$\|\mu\| = \frac{1}{N+1}\sum_{i=0}^{N} f_i, \qquad (2)$$

where f_i is the function being reconstructed/interpolated.

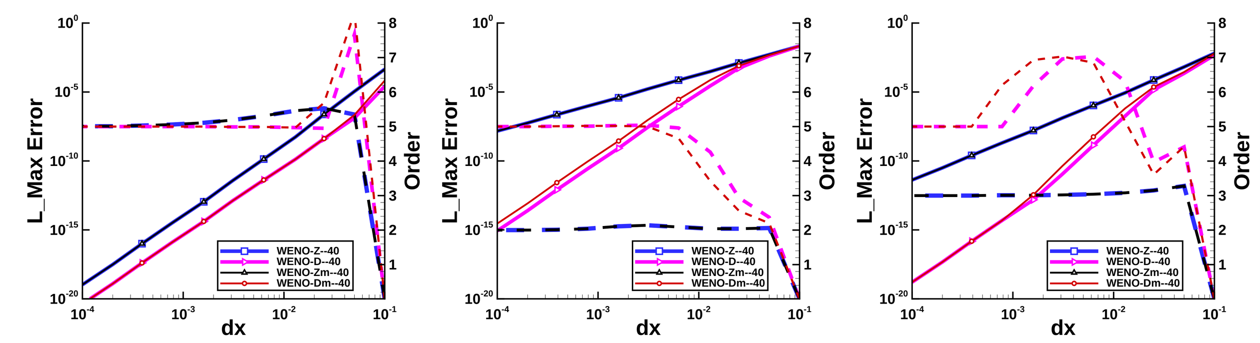
- In the scalar case $f_i = |q(x_i)|$, where $q(x)$ can be the conservative variables $\mathbf{Q}$ or flux $\mathbf{F}$ used in a WENO finite volume or difference scheme respectively.
- In the system case (e.g. the Euler equation), if $q^{(m)}(x)$ is the m-th component of the characteristic flux variables $\mathbf{LF}$, where the flux $\mathbf{F}$ is projected onto the characteristic fields spanned by M left Roe-averaged eigenvectors $\mathbf{L}$ at $x_{i+\frac{1}{2}}$, then $f_i = \frac{1}{5}\sum_{j=-2}^{2}|q^{(m)}(x_{i+j})|, m = 1, \ldots, M$.

Critical Points

Consider the following test function

$$f(x) = \kappa x^{n+1} e^{\lambda x}, \quad x \in [-1, 1], \quad \lambda = 0.75. \qquad (3)$$

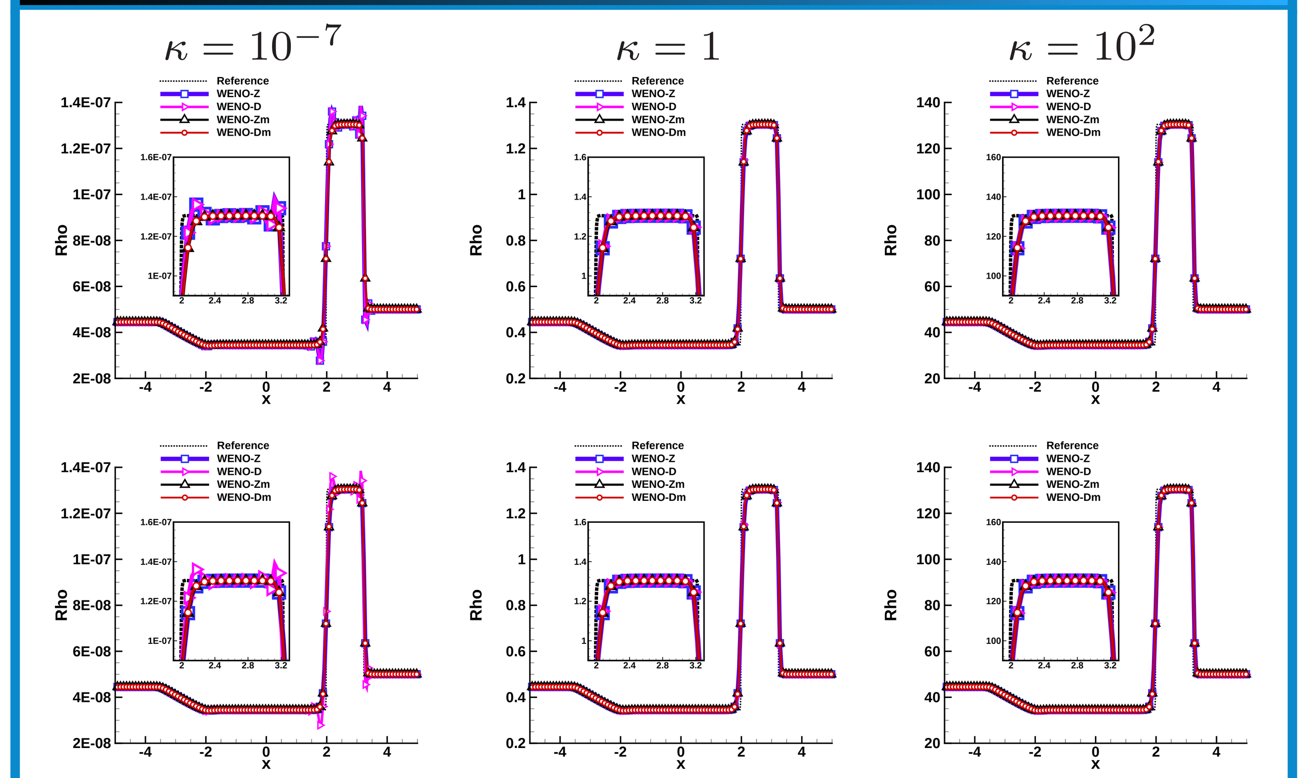
This function has a critical point of order $n_{cp} = n$ at $x = 0$ [2]. We will only present the L^∞ error (solid line) and the order of accuracy (dashed line) of the four WENO schemes with a constant sensitivity parameters ($\varepsilon = 10^{-40}$) and with ($\kappa = 1$ and $n_{cp} = 1, 2, 3$) in the figure and with ($\kappa = 10^{-7}$ and $n_{cp} = 3$) in Table.



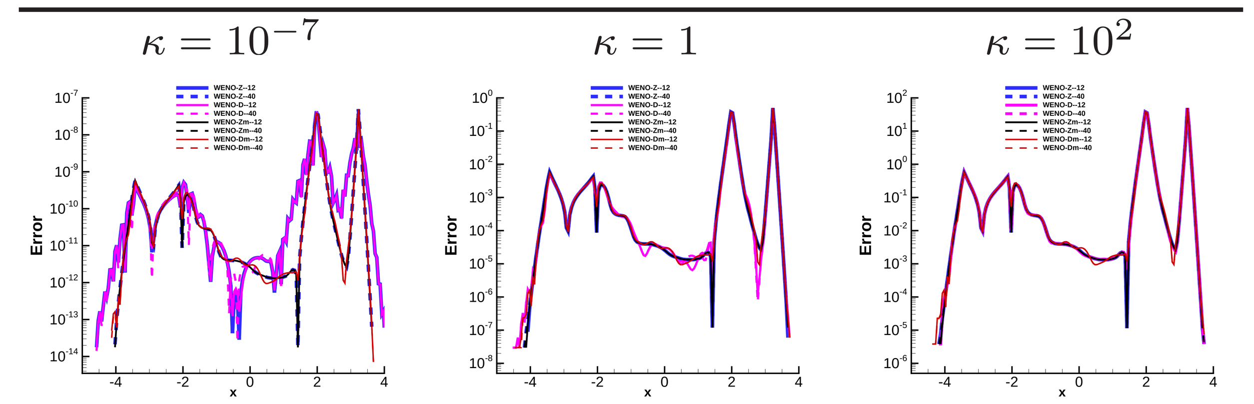
These results show that the WENO-D/Dm schemes achieve the optimal fifth order. However, The WENO-Z/Zm schemes can attain only a much reduced order (the second order with $n_{cp} = 2$ and the third order with $n_{cp} = 3$). To summarize, the WENO-D/Dm schemes satisfy the Cp-property. They also have a better accuracy in general than the WENO-Z/Zm schemes.

	$\varepsilon = 10^{-12}$		$\varepsilon = 10^{-40}$	
N	L_∞ error	Order	L_∞ error	Order
160	2.0E−14	—	6.5E−14	—
320	1.6E−17	10.3	5.6E−16	6.9
640	4.9E−19	5.0	4.3E−18	7.0
1280	1.5E−20	5.0	3.5E−20	6.9
2560	4.8E−22	5.0	4.8E−22	6.2
5120	1.5E−23	5.0	1.5E−23	5.0
10240	4.7E−25	5.0	4.7E−25	5.0
20480	1.5E−26	5.0	1.5E−26	5.0

Lax Problem ($N = 200$)

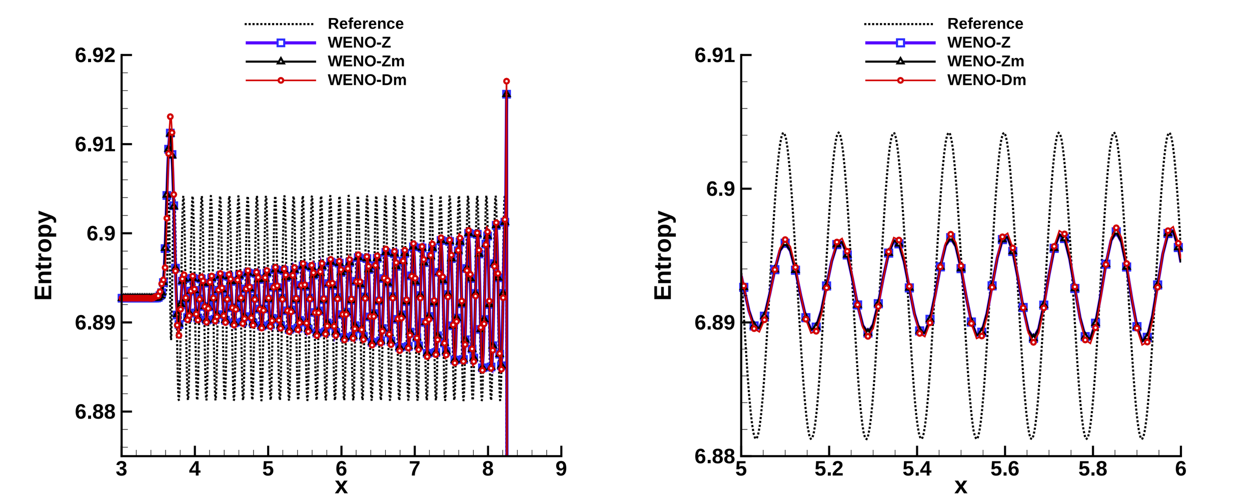


Lax problem [4] with $\varepsilon = 10^{-12}$ (top) and $\varepsilon = 10^{-40}$ (bottom).

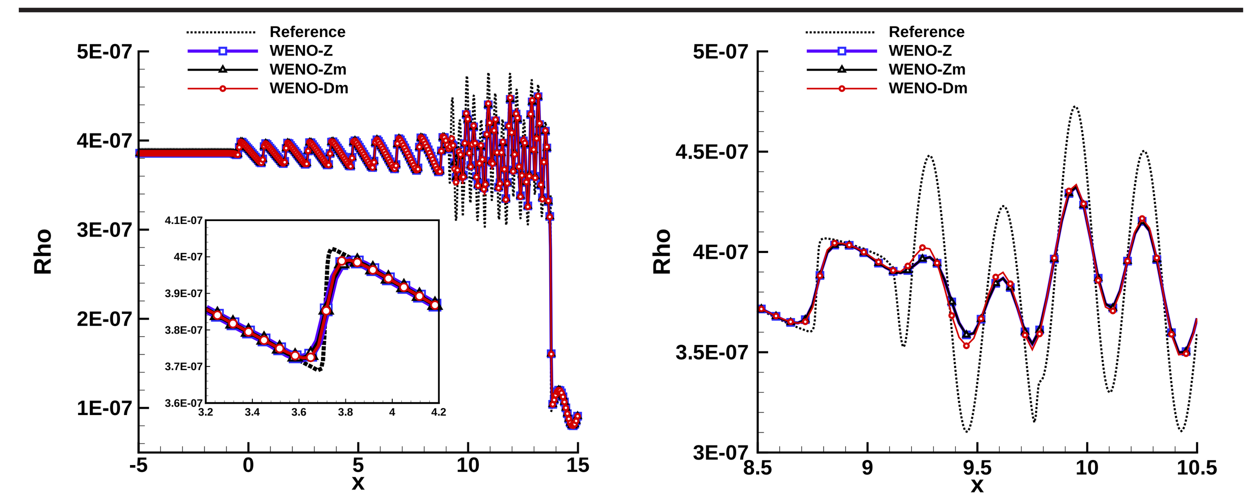


Point-wise errors of Lax problem [4] with $\varepsilon = 10^{-12}, 10^{-40}$.

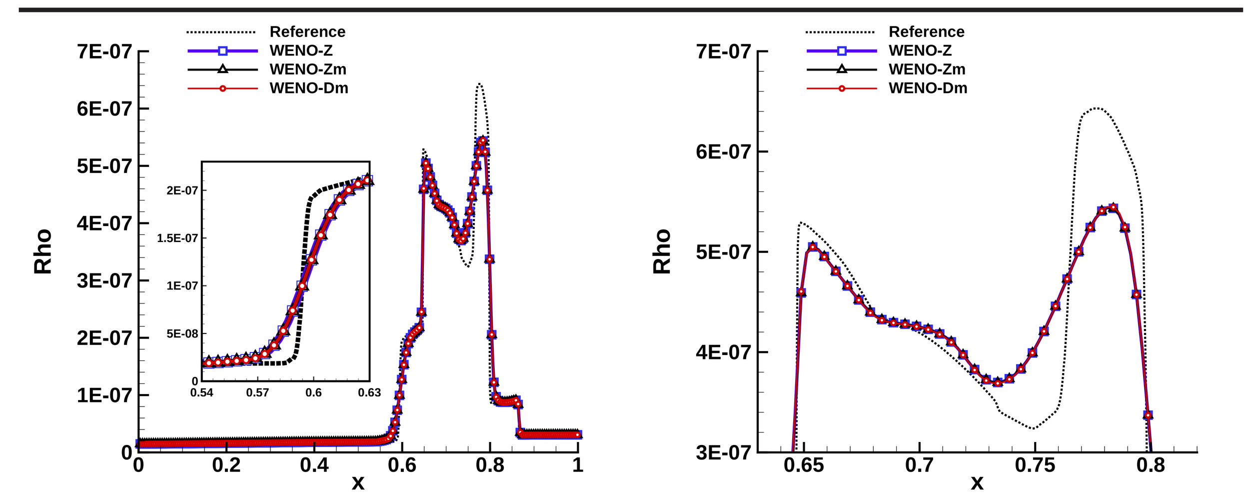
1D Examples ($\kappa = 10^{-7}, \varepsilon = 10^{-40}$)



Shock-Entropy Problem [5] with $N = 1700$.

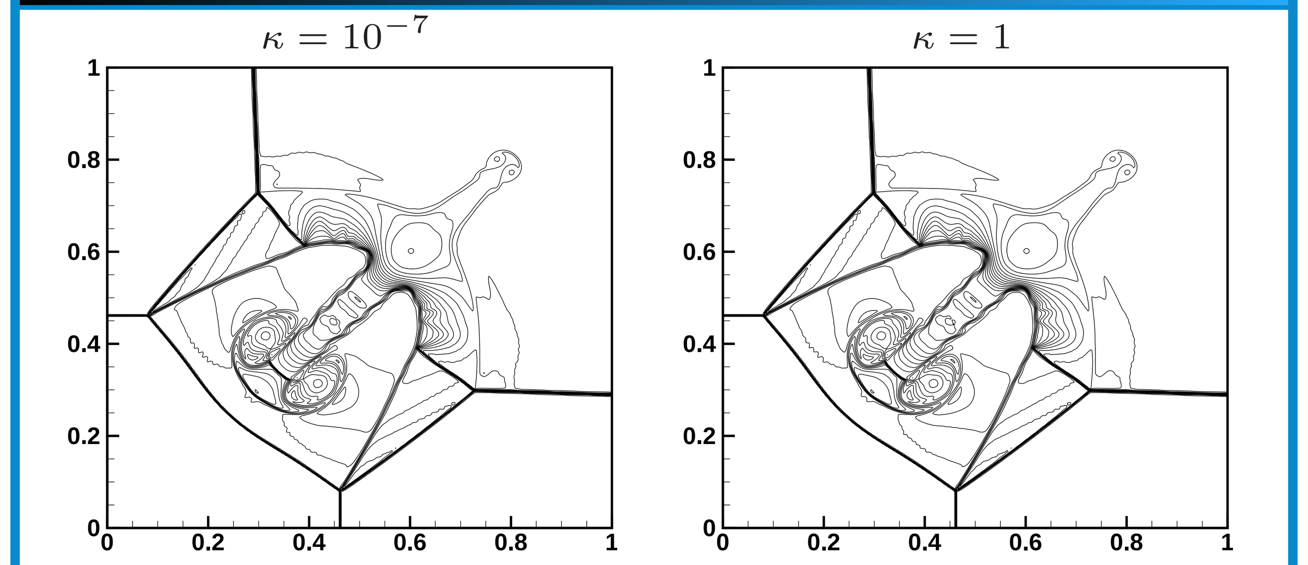


Shock-Density Problem [1] with $N = 600$.

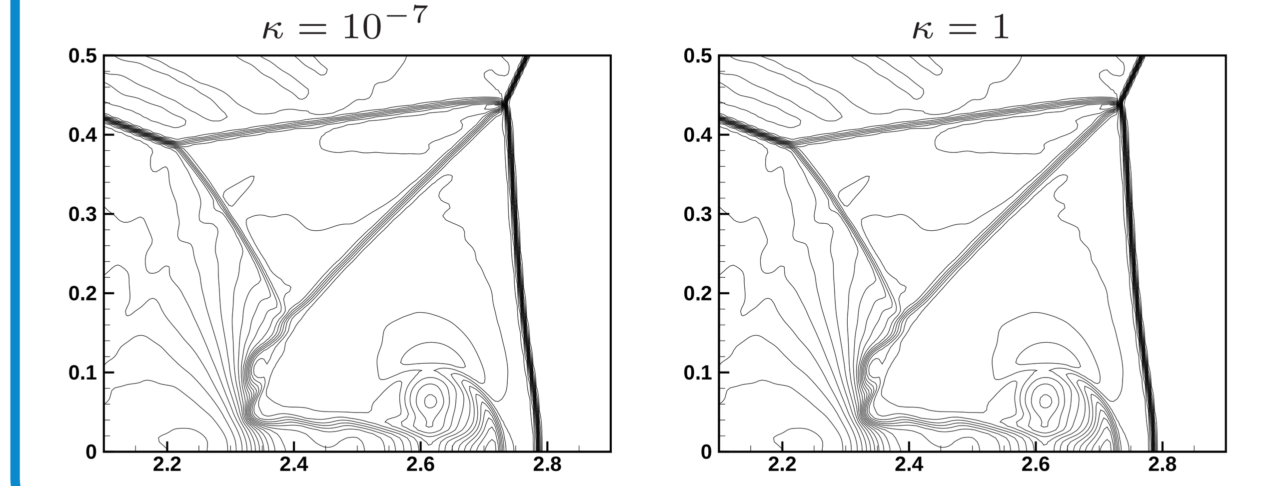


Blastwave Problem [7] with $N = 400$.

2D Examples



Riemann Problem (Configuration 3) [3] with $N \times M = 400 \times 400$.



Double Mach Reflection [7] with $N \times M = 800 \times 200$.

Conclusion

In this study, we have proposed a simple and effective modification of the Z-type nonlinear weights in a high order weighted essentially non-oscillatory (WENO) finite difference scheme that satisfies both the Cp-property and the Si-property. By introducing the descaling function, which is a global average of the function values, in the definition of the nonlinear weights in the WENO-Dm scheme, the scaling dependencies of both the modifier function and the sensitivity parameter are removed. The Cp-property of the WENO-Z/D/Zm/Dm schemes are verified with different scaling factors and sensitivity parameters. The Si-property of the WENO-Z/D/Zm/Dm schemes are illustrated via several one- and two-dimensional shock-tube benchmark problems with small and large scaling factors. The results show that only the WENO-Zm/Dm schemes are able to maintain the ENO property at a discontinuity regardless of the scaling.

To summarize, the WENO-Dm scheme satisfies both the Cp-property and the Si-property simultaneously.

Acknowledgement

The authors would like to acknowledge the funding support of this research by the Shandong Provincial Natural Science Foundation (ZR2017MA016). The author (Don) also likes to thank the Ocean University of China for providing the startup funding (201712011).

References

[1] R. Borges, M. Carmona, B. Costa & W.S. Don, JCP, 227 (2008) 3191–3211.
[2] W.S. Don & R. Borges, JCP, 250 (2013) 247–372.
[3] P.D. Lax & X.-D. Liu, SISC, 19 (1998) 319–340.
[4] P.D. Lax, PAM, 7 (1954) 159–193.
[5] V.A. Titarev & E.F. Toro, JCP, 201 (2004) 238–260.
[6] Y. Wang, B.-S. Wang & W.S. Don, JSC, 81 (2019) 1329–1358.
[7] P. Woodward & P. Colella, JCP, 54 (1984) 115–173.