WENO-Z Finite Difference Scheme for Hyperbolic Conservative Equations

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Introduction

The main objective of this work is to study the fifth-order WENO finite difference scheme which is capable of capturing sharp discontinuities in an essentially non-oscillatory manner. It uses a convex combination of three candidate stencils, each producing a second order accurate flux, to obtain fifth- order accuracy and an essentially non-oscillatory shock transition. Time discretization can be implemented by the third-order TVD Runge-Kutta method.

In our numerical experiments, the WENO-Z scheme with Lax-Friedrichs splitting is applied to solve the hyperbolic conservative equations included the wave equation and Euler equations. The results demonstrate the high order accuracy of the WENO-Z scheme with essentially free oscillation.

WENO-Z Scheme

We consider a scalar hyperbolic conservation law equation in one dimension:

$$\frac{\partial Q}{\partial t}+\frac{\partial F(Q)}{\partial x}=0,\ x\in[0,2\pi].$$

The spatial derivative $F(Q)_x$ is approximated by a conservative flux difference:

$$F(Q)_x|_{x=xi}\approx\frac{1}{\Delta x}(\widehat{F}_{i+\frac{1}{2}}-\widehat{F}_{i-\frac{1}{2}}).$$

The numerical flux $\widehat{F}_{i+\frac{1}{2}}$ is computed through the neighboring point values $F_j = F(Q)_j$. As for the fifth-order WENO difference scheme, we shall compute three reconstructed polynomials firstly

$$\widehat{F}^{(r)}_{i+\frac{1}{2}}=\sum_{j=0}^{2}d_{rj}F_{i-r+j}, r=0,1,2$$

where d_{rj} is obtained from a construction of polynomial. Specifically,

$$\begin{aligned}\widehat{F}^{(0)}_{i+\frac{1}{2}}&=\tfrac{1}{3}F_{i-2}-\tfrac{7}{6}F_{i-1}+\tfrac{11}{6}F_i\\ \widehat{F}^{(1)}_{i+\frac{1}{2}}&=-\tfrac{1}{6}F_{i-1}+\tfrac{5}{6}F_i+\tfrac{1}{3}F_{i+1}\\ \widehat{F}^{(2)}_{i+\frac{1}{2}}&=\tfrac{1}{3}F_i+\tfrac{5}{6}F_{i+1}-\tfrac{1}{6}F_{i+2}\end{aligned}$$

The fifth-order WENO flux is a convex combination of all these reconstruction polynomials:

$$\widehat{F}_{i+\frac{1}{2}}=\sum_{j=0}^{2}w_j\widehat{F}^{(j)}_{i+\frac{1}{2}}, w_j\geq 0, \sum_{j=0}^{2}w_j=1.$$

where the nonlinear weights w_j are defined by WENO-Z difference scheme.

WENO-Z scheme

In the fifth order WENO-Z schemes, the nonlinear weights w_j are defined as

$$w_j=\frac{\alpha_j}{\Sigma_{s=0}^{2}\alpha_s},\ \alpha_j=c_j(1+(\frac{\tau_2}{\beta_j+\epsilon})^p),\ \tau_2=|\beta_0-\beta_2|$$

where c_j are the optimal linear weights which are given as:

$$c_0=\frac{3}{10},\ c_1=\frac{3}{5},\ c_2=\frac{1}{10}.$$

β_j (j=0,1,2) are the smooth indicators, are defined as:

$$\beta_j=\Sigma_{k=1}^{2}\int_{I_i}\Delta x_j^{2k-1}(\frac{d^k p_j(x)}{dx^k})^2dx$$

Specifically,

$$\begin{aligned}\beta_0&=\tfrac{13}{12}(f_{i-2}-2f_{i-1}+f_i)^2+\tfrac{1}{4}(f_{i-2}-4f_{i-1}+3f_i)^2\\ \beta_1&=\tfrac{13}{12}(f_{i-1}-2f_i+f_{i+1})^2+\tfrac{1}{4}(f_{i-1}-f_{i+1})^2\\ \beta_2&=\tfrac{13}{12}(f_i-2f_{i+1}+f_{i+2})^2+\tfrac{1}{4}(3f_i-4f_{i+1}+f_{i+2})^2\end{aligned}$$

WENO-Z scheme is capable of capturing sharp discontinuities in an essentially non-oscillatory manner resolving high frequency waves accurately with less dissipation and higher resolution than WENO-JS scheme for a larger class of nonlinear hyperbolic PDEs.

An upwind mechanism, essential for the stability of the scheme, can be realized by a global "flux splitting". Lax-Friedrichs (LF) splitting is applied here, which is defined as:

$$f^{\pm}(Q)=\frac{1}{2}(f(Q)\pm\alpha Q)$$

where $\alpha = max_Q|f'(Q)|$. As for time discretization, we used the third-order TVD Runge-Kutta scheme with CFL=0.45.

Euler equation

The Euler equation is:

$$\frac{\partial Q}{\partial t}+\frac{\partial F(Q)}{\partial x}=0$$

where

$$Q=\begin{bmatrix}\rho\\ \rho u\\ E\end{bmatrix},\quad F(Q)=\begin{bmatrix}\rho u\\ \rho u^2+p\\ u(E+p)\end{bmatrix}\qquad(1)$$

with equation of state:

$$p=(\gamma-1)(E-\frac{1}{2}\rho u^2)$$

where ρ is the density, p is the pressure, u is the velocity, E is the total energy and $\gamma = 1.4$ is the ratio of specific heat.

Wave equation

Table 1: L^{∞} error of the fifth-order WENO-Z finite difference scheme at t=2.

N	$L^{\infty}error$	$Order$
25	3.2529E-4	–
50	9.9052E-6	5.0374
100	3.0520E-7	5.0203
200	9.1242E-9	5.0639

We solve the linear wave equation with the following initial conditions:

$$Q(x,0)=\sin(x), x=[0,2\pi]$$

In Table 1, numerical results show that the fifth order accuracy can be achieved in the wave equation.

Three kind of Riemann Problems

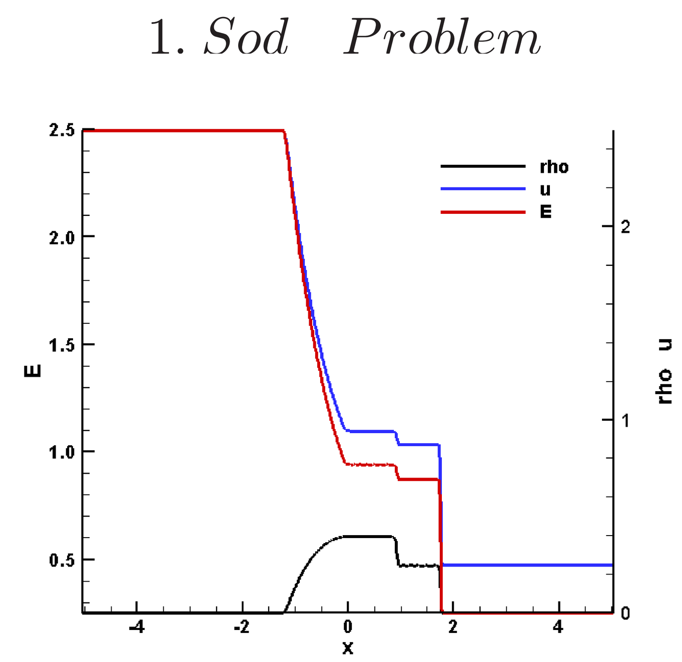

Figure 1: Sod shock tube problem as computed by fifth order finite difference WENO-Z scheme at $t = 2$ with $N = 800$ cells.

$$(\rho_l,u_l,p_l)=(1,0,1)$$
$$(\rho_r,u_r,p_r)=(0.125,0,0.1)$$

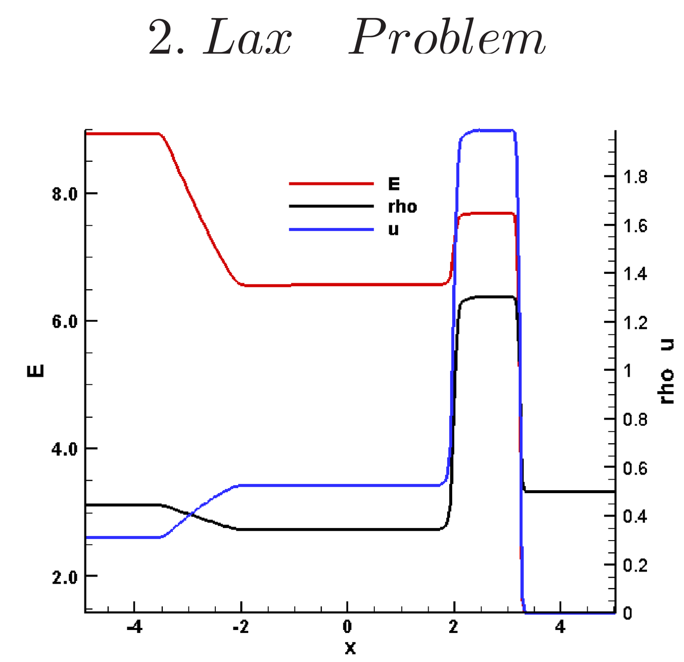

Figure 2: Lax shock tube problem as computed by fifth order finite difference WENO-Z scheme at $t = 1.3$ with $N = 800$ cells.

$$(\rho_l,u_l,p_l)=(0.445,0.698,3.528)$$
$$(\rho_r,u_r,p_r)=(0.5,0,0.571)$$

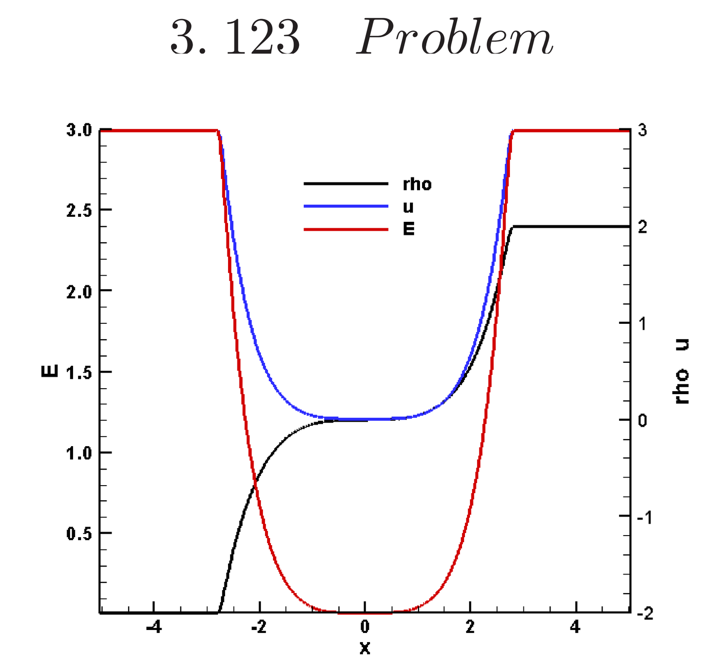

Figure 3: 123 shock tube problem as computed by fifth order finite difference WENO-Z scheme at $t = 1$ with $N = 800$ cells.

$$(\rho_l,u_l,p_l)=(1.0,-2.0,0.4)$$
$$(\rho_r,u_r,p_r)=(1.0,2.0,0.4)$$

Despite the small oscillation near the discontinuity, the performance of the scheme shows the effectivity and robustness. The results of the three kind of Riemann questions demonstrate the high order accuracy with essentially free oscillation.

Future Work

- Extension of the finite difference WENO-Z scheme with characteristic decomposition.
- Extension of finite difference WENO-Z scheme to solve the 2D and 3D Euler equations.

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