Forth Order Central-Upwind WENO Scheme with Modified Ideal Weights

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Introduction

In this work, we modify the ideal weights of the classical WENO-Z scheme to be more central and have better performance on computing problems with high frequency parts.

WENO-Z Scheme

The $(2r-1)$ degree polynomial approximation $\hat{f}_{i\pm\frac{1}{2}} = h_{i\pm\frac{1}{2}} + O(\Delta x^{2r-1})$ is built through a convex combination of the interpolated values $\hat{f}^k(x_{i\pm\frac{1}{2}})$, in which the $(r-1)$ degree interpolation polynomial $\hat{f}^k(x)$, together with the nonlinear weight ω_k, are defined in each one of the substencils S_k:

$$\hat{f}_{i\pm\frac{1}{2}} = \sum_{k=0}^{r-1} \omega_k \hat{f}^k(x_{i\pm\frac{1}{2}}). \tag{1}$$

$$\hat{f}^k(x_{i+\frac{1}{2}}) = \hat{f}^k_{i+\frac{1}{2}} = \sum_{j=0}^{r-1} c_{k,j} f_{i+j+k-(r-1)}, \tag{2}$$

$$\omega_k = \frac{\alpha_k}{\sum_{j=0}^{r-1}\alpha_j}, \alpha_k = d_k\left(1+\left(\frac{\tau_{2r-1}}{\beta_k+\varepsilon}\right)^p\right), \tag{3}$$

the (lower order local) smoothness indicators β_k are given by

$$\beta_k = \sum_{l=1}^{r-1} \Delta x^{2l-1} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\frac{d^l}{dx^l}\hat{f}^k(x)\right)^2 dx.$$

For fifth order WENO-Z scheme [1], the optimal order global smoothness indicator $\tau_5 = |\beta_0 - \beta_2|$, the ideal weights $d_0 = \frac{1}{10}, d_1 = \frac{3}{5}, d_2 = \frac{3}{10}$ and $p = 2, \varepsilon = 10^{-12}$.

Modified Ideal Weights

To make the WENO-Z scheme more central, we introduce the modified ideal weights:

$$d_0 = \phi, \quad d_1 = \frac{1}{2} + \phi, \quad d_2 = \frac{1}{2} - 2\phi, \tag{4}$$

where

$$\phi = \eta + \left(\frac{1}{10} - \eta\right)\left(1 - \frac{1}{(1+\tau_5)^2}\right), \tag{5}$$

with a user parameter $\eta \in [0, \frac{1}{10}]$, which is used to control the dispassion of the WENO scheme. When $\eta = \frac{1}{10}$, we obtain the classical ideal weights $d_0 = \frac{1}{10}, d_1 = \frac{3}{5}, d_2 = \frac{3}{10}$ and the scheme is the fifth-order WENO-Z scheme with the most dispassion. When $\eta = 0$ for smooth problem, we have $\phi = 0$, then the ideal weights become $d_0 = 0, d_1 = d_2 = \frac{1}{2}$ and the scheme is fourth-order central scheme with the less dispassion. For the smooth problem, the scheme with an increasing η has the increasing dispassion. For the non-smooth problem, the term $\frac{1}{(1+\tau_5)^2} \approx 0$ and $\phi \approx \frac{1}{10}$. The classical fifth-order WENO-Z scheme is used for the non-smooth problem. In the following tests, we use $\eta = \frac{1}{40}$.

Remark of the Scheme

For simplicity, we use "-I" to denote the WENO-Z5 scheme with the modified ideal weights. The third order TVD Runge-Kutta method with CFL $= 0.45$ is used for time integration. Here, we consider the 1D shock-entropy problem, the shock-density problem [1, 3] and blast wave problem [4], the 2D Riemann problem [2], isentropic vortex problem [5], double Mach reflection (DMR) problem and forward facing step (FFS) problem [4].

Accuracy Tests

Consider the 1D wave equation $Q_t + Q_x = 0$ with the initial condition $Q(x) = \sin x, x \in [0, 2\pi]$, the periodical boundary condition and final time $t = 2$ ($\Delta t = \Delta x^{4/3}$).

N	L^∞ Error	Order	L^2 Error	Order
10	7.2E-3	—	1.3E-2	—
20	4.8E-4	3.92	8.6E-4	3.91
40	3.0E-5	3.97	5.4E-5	3.99
80	1.9E-6	4.00	3.4E-6	4.00
160	1.2E-7	4.00	2.1E-7	4.00
320	7.4E-9	4.00	1.3E-8	4.00

Consider the well-known 2D inviscid isentropic vortex problem with the exact boundary condition and the final time $t = 1$ ($\Delta t = \Delta x^{4/3} = \Delta y^{4/3}$).

N^2	L^∞ Error	Order	L^2 Error	Order
10^2	1.7E-1	—	2.7E-2	—
20^2	2.4E-2	2.78	3.4E-3	2.96
40^2	4.2E-3	2.53	4.3E-4	3.01
80^2	6.5E-4	2.69	4.0E-5	3.41
160^2	3.4E-5	4.21	1.7E-6	4.59
320^2	9.3E-7	5.23	8.5E-8	4.29

Future Work

Extend this scheme to sixth and higher order cases.

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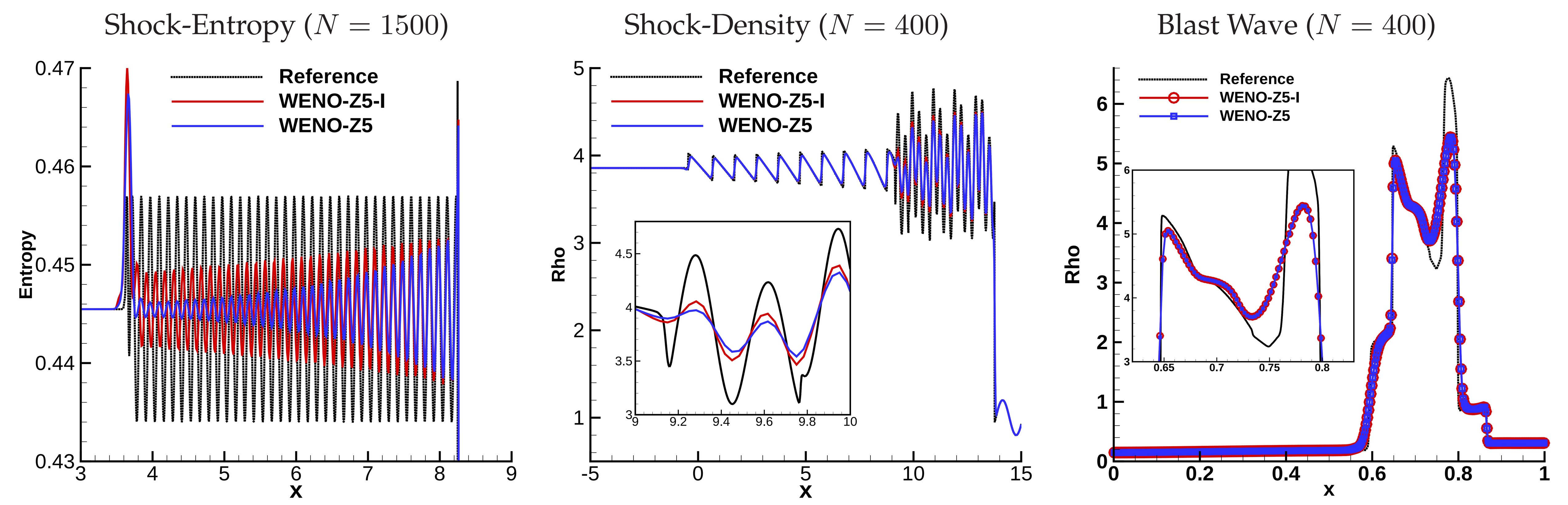
1D Euler Equations



2D Euler Equations