An Improved Alternative WENO Scheme

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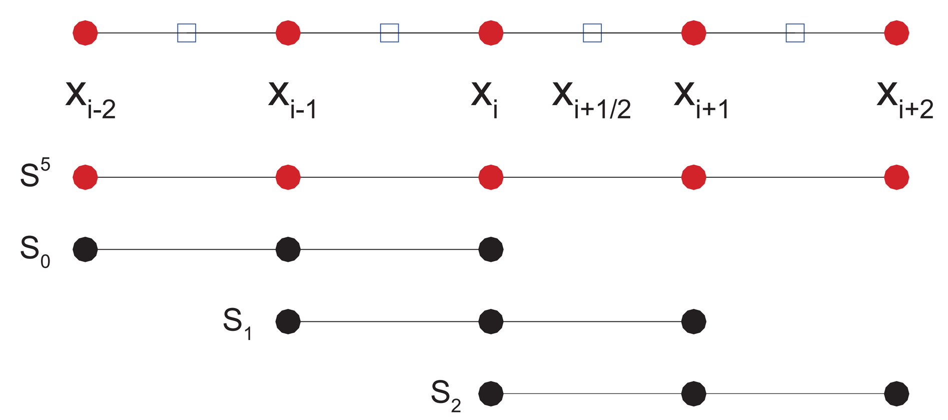
Introduction

In this formulation, the WENO interpolation of the solution and its derivatives are used to directly construct the numerical flux, instead of the usual practice of reconstructing the flux functions. This means that arbitrary monotone fluxes can be used in this framework, while the traditional practice of reconstructing flux functions can be applied only to smooth flux splitting. In this work, we improve the alternative WENO scheme by WENO-Z weights and verify the accuracy of the improved alternative WENO scheme.

WENO Interpolation



Given the point values $u_i = u(x_i)$ of a function $u(x)$, we need to find an approximation of $u(x)$ at the half nodes $x_{i+\frac{1}{2}}$ using the polynomial interpolation. Find a unique polynomial, $p(x)$, which interpolates the function $u(x)$, that is, $p(x_j) = u_j$, at the mesh points x_j, in the stencil.

For the small stencils,

$$\begin{aligned} u^{(0)}_{i+\frac{1}{2}} &= \frac{3}{8}u_{i-2} - \frac{5}{4}u_{i-1} + \frac{15}{8}u_i, \\ u^{(1)}_{i+\frac{1}{2}} &= -\frac{1}{8}u_{i-1} + \frac{3}{4}u_i + \frac{3}{8}u_{i+1}, \\ u^{(2)}_{i+\frac{1}{2}} &= \frac{3}{8}u_i + \frac{3}{4}u_{i+1} - \frac{1}{8}u_{i+2}. \end{aligned} \tag{1}$$

For the big stencil S,

$$u_{i+\frac{1}{2}} = \frac{3}{128}u_{i-2} - \frac{5}{12}u_{i-1} + \frac{45}{64}u_i + \frac{15}{32}u_{i+1} - \frac{5}{128}u_{i+2}. \tag{2}$$

For smooth case, we wish

$$u_{i+\frac{1}{2}} = \sum_{k=0}^{2} d_k u^{(k)}_{i+\frac{1}{2}}, \tag{3}$$

where d_k are the *linear weights*,

$$d_0 = \frac{1}{16}, \quad d_1 = \frac{5}{8}, \quad d_2 = \frac{5}{16}. \tag{4}$$

Change the linear weights d_k into the nonlinear weights ω_k,

$$\omega_k = \frac{\alpha_k}{\sum_{s=0}^{2}\alpha_s}, \quad k = 0, 1, 2, \tag{5}$$

$$\alpha_k^{JS} = \frac{d_k}{(\beta_k + \epsilon)^p}, \quad \text{or} \quad \alpha_k^{Z} = d_k\left(1 + \left(\frac{\tau_5}{\beta_k + \epsilon}\right)^p\right), \tag{6}$$

where β_k are *smoothness indicators* of the stencil S_k, which measures the smoothness of $u(x)$ in stencil S_k, p is the power parameter, ϵ is the parameter to avoid the denominator to be zero, and $\tau_5 = |\beta_0 - \beta_2|$. Usually, $p = 2, \epsilon = 10^{-12}$.

WENO Interpolation

Here, the smoothness indicators are given by

$$\begin{aligned} \beta_0 &= \frac{13}{12}(u_{i-2} - 2u_{i-1} + u_i)^2 + \frac{1}{4}(u_{i-2} - 4u_{i-1} + 3u_i)^2, \\ \beta_1 &= \frac{13}{12}(u_{i-1} - 2u_i + u_{i+1})^2 + \frac{1}{4}(u_{i-1} - u_{i+1})^2, \\ \beta_2 &= \frac{13}{12}(u_i - 2u_{i+1} + u_{i+2})^2 + \frac{1}{4}(3u_i - 4u_{i+1} + 3u_{i+2})^2. \end{aligned}$$

Thus, we get the WENO interpolation of $u(x)$ at $x_{i+\frac{1}{2}}$ as

$$u_{i+\frac{1}{2}} = \sum_{k=0}^{2} \omega_k u^{(k)}_{i+\frac{1}{2}}. \tag{7}$$

We denote $u^-_{i+\frac{1}{2}}$ and $u^+_{i+\frac{1}{2}}$, respectively, for the big stencils $S = \{x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}\}$ and $S = \{x_{i-1}, x_i, x_{i+1}, x_{i+2}, x_{i+3}\}$. In fact, the process to obtain $u^+_{i+\frac{1}{2}}$ is mirror-symmetric to that for $u^-_{i+\frac{1}{2}}$, with respect to the target point $x_{i+\frac{1}{2}}$.

Construction of the Scheme

Assuming that $f(u)$ is a smooth function of u, we would like to find a consistent numerical flux function, $\hat{f}_{i+\frac{1}{2}} = \hat{f}(u_{i-2}, \cdots, u_{i+3})$, such that the flux difference approximates the derivative $f(u(x))_x$ to 5-th order accuracy

$$\frac{1}{\triangle x}(\hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}}) = f(u(x))_x\big|_{x_i} + O(\triangle x^5). \tag{8}$$

Here, we can use

$$\hat{f}_{i+\frac{1}{2}} = f_{i+\frac{1}{2}} - \frac{1}{24}\triangle x^2 f_{xx}\big|_{i+\frac{1}{2}} + \frac{7}{5760}\triangle x^4 f_{xxxx}\big|_{i+\frac{1}{2}}. \tag{9}$$

The first term of the numerical flux is approximated by $f_{i+\frac{1}{2}} = h(u^-_{i+\frac{1}{2}}, u^+_{i+\frac{1}{2}})$ with the values $u^{\pm}_{i+\frac{1}{2}}$ obtained by the WENO interpolation. The two-argument function h is a monotone flux, such as Godunov flux, Engquist-Osher flux, Lax-Friedrichs flux, HLLC flux. We approximate the remaining terms $f_{xx}|_{i+\frac{1}{2}}$, $f_{xxxx}|_{i+\frac{1}{2}}$, respectively, by simple central approximation,

$$\frac{1}{48\triangle x^2}(-5f_{i-2} + 39f_{i-1} - 34f_i - 34f_{i+1} + 39f_{i+2} - 5f_{i+3}),$$

$$\frac{1}{2\triangle x^4}(f_{i-2} - 3f_{i-1} + 2f_i + 2f_{i+1} - 3f_{i+2} + f_{i+3}).$$

Remark of the Scheme

Together with third order TVD Runge-Kutta method, Roe eigensystem, Lax-Friedrichs Riemann solver and HLLC Riemann solver, the scheme is used for solving the hyperbolic conservation systems, such as Euler equations in this work. For the higher dimensional problems, the scheme is applied in the x- and y- directions respectively.

Accuracy of the Alternative Scheme

Table 1: Accuracy on $u_t + u_x = 0, x \in [-1, 1]$ with periodic boundary condition and $u(x, 0) = \sin(\pi x)$ at $t = 2$.

	WENO-JS-A		WENO-Z-A	
N	L^∞ error	Order	L^∞ error	Order
10	4.1e-2	—	9.6e-3	—
20	2.1e-3	4.29	3.0e-4	4.99
40	7.3e-5	4.84	9.5e-6	4.99
80	2.3e-6	4.98	3.0e-7	5.00
160	7.0e-8	5.05	9.4e-9	5.00

Table 2: Accuracy at the first-order critical point $x = 0$ of $f(x) = x^3 + \cos(x)$ with $\epsilon = 10^{-40}, p = 2$ and 64 digits.

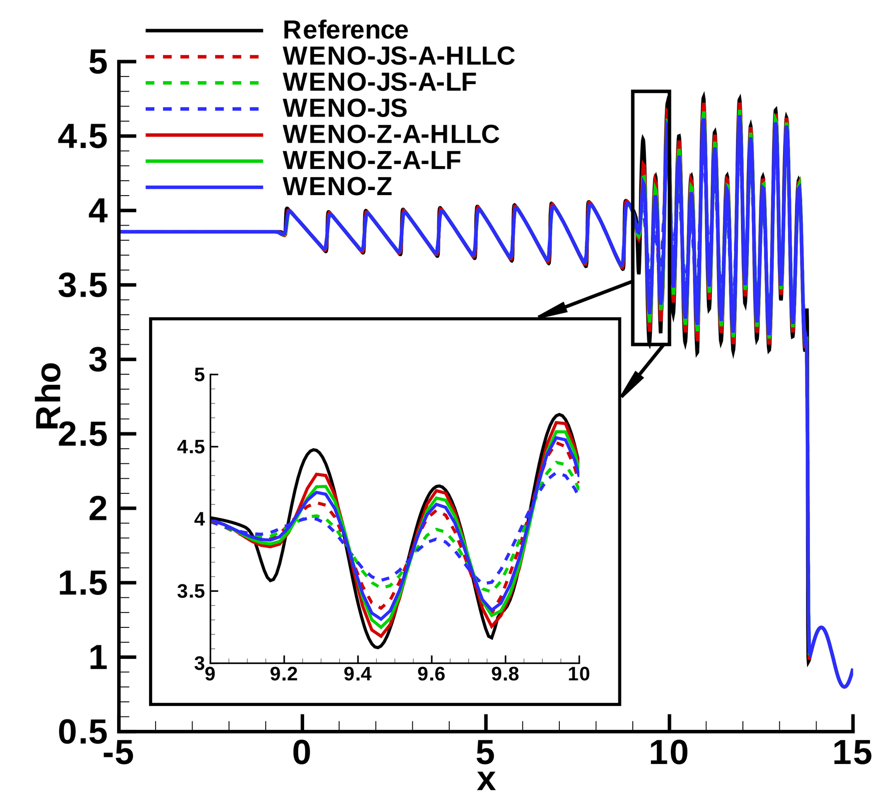
	WENO-JS-A		WENO-Z-A	
$\triangle x$	L^∞ error	Order	L^∞ error	Order
1.00e-3	1.3e-9	—	1.6e-13	—
5.00e-4	1.6e-10	2.95	5.8e-15	4.80
2.50e-4	2.1e-11	2.98	1.9e-16	4.91
1.25e-4	1.6e-12	2.99	6.2e-18	4.96
6.25e-5	3.3e-13	3.00	2.0e-19	4.98

1D Euler Equations

Table 3: Accuracy on 1D Euler equations by the WENO-Z-A scheme with different Riemann solvers, where the initial condition is $(\rho, u, P) = (1 + 0.1\sin(\pi x), 1, 1)$.

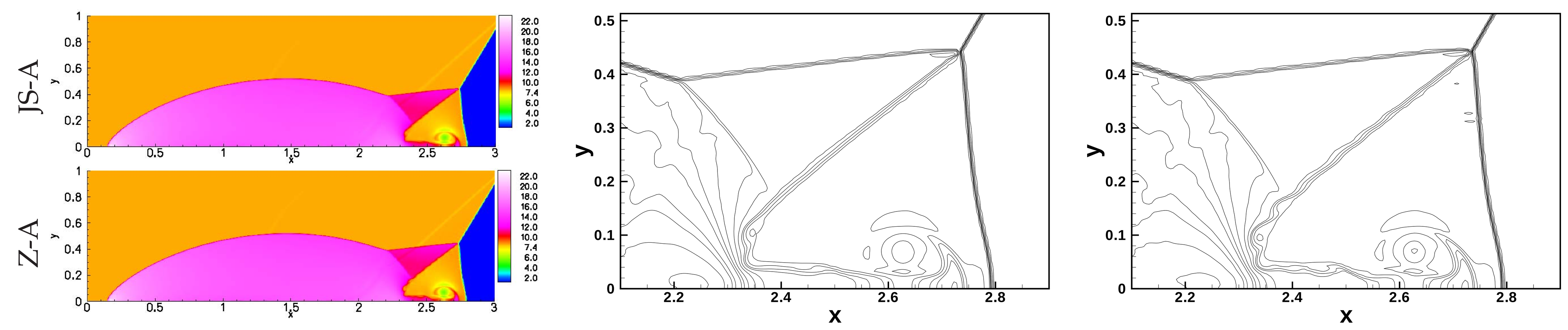
	Lax-Friedrichs solver		HLLC solver	
N	L^∞ error	Order	L^∞ error	Order
10	1.5e-3	—	7.9e-4	—
20	5.0e-5	4.93	2.2e-5	5.13
40	1.6e-6	4.99	7.0e-7	4.99
80	4.9e-8	4.98	2.2e-8	4.99
160	1.5e-9	5.00	6.9e-10	5.00

Consider the extended Shu-Osher problem ($N = 800$) to show that the improved alternative WENO scheme is more accurate than the other schemes.



2D Problem

For 2D problem, take the DMR problem (800×200) as an example.



The colored contour of density (Left) and their "zoomed-in" graphs (Middle: WENO-JS-A, Right: WENO-Z-A) show that the WENO-Z-A scheme captures the shocks more accurately than the WENO-JS-A scheme.

Future Work

Research on free-stream preserving alternative WENO schemes and hybrid schemes on curvilinear meshes.

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References

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