Radial Basis Function Based Shock Detection Method

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Introduction

In this study, the radial basis function (RBF) method, together with the Tukey's boxplot method and the domain segmentation technique, is extended to serve as a novel shock detection algorithm for solving the Euler equations. The applicability and performance of the RBF shock detector in the hybrid Compact-WENO scheme in terms of accuracy, robustness, efficiency, resolution and other implementation issues are given. Several one- and two-dimensional benchmark problems in shocked flow demonstrate that the proposed hybrid scheme can reach a speedup of the CPU times by a factor up to 2-3 compared with the pure fifth order WENO-Z scheme.

RBF Approximation

The multi-quadric RBFs are defined by

$$\phi_j(x) = \sqrt{(x-x_j)^2 + \epsilon_j^2},\ x_j \in \mathbf{X},\ x \in \mathbf{\Omega},\ j = 1, \cdots, N, \quad (1)$$

where $\mathbf{X} \subset \mathbf{\Omega}$ is a set of centers and ϵ_j are corresponding shape parameters.

The MQ-RBF approximation $g(x)$ and its derivative for a real-valued function $f(x)$ are given by

$$g(x) = \sum_{j=1}^{N} \lambda_j \phi_j(x), \quad g'(x) = \sum_{j=1}^{N} \lambda_j \frac{x - x_j}{\phi_j(x)}, \quad (2)$$

where λ_j are the expansion coefficients which can be obtained by enforcing the interpolation conditions. By defining the interpolation matrix $\mathbf{M}$ and the differentiation matrix $\mathbf{D}$,

$$M_{ij} = \sqrt{(x_i - x_j)^2 + \epsilon_j^2}, \quad D_{ij} = (x_i - x_j)/M_{ij}, \quad (3)$$

the expansion coefficients vector $\vec{\lambda}$ and the derivative vector $\vec{g}'$ are given by

$$\vec{\lambda} = \mathbf{M}^{-1}\vec{f}, \quad \vec{g}' = \mathbf{D}\vec{\lambda}. \quad (4)$$

Here, we assume $D_{jj} = 0$ if $\epsilon_j = 0$.

Note that $\mathbf{M}$ is a symmetric Toeplitz matrix on a uniformly spaced mesh with a constant shape parameter $\epsilon > 0$, which can be solved fast by the $O(N^2)$ recursive Levinson-Durin algorithm. The ill-condition of the interpolation matrix $\mathbf{M}$ can be alleviated by domain rescaling (Jung *et al.* 09).

Non-Smooth Centers

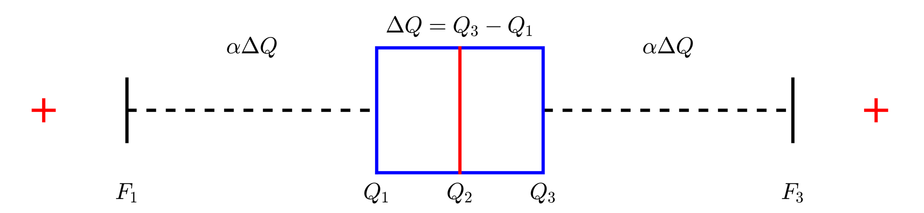
The non-smooth centers $\mathbf{S}$ can be defined as

$$\mathbf{S} = \{x_i | d_i \geq \eta,\ d_i = |\vec{g}'_i|^2,\ x_i \in \mathbf{X},\ i = 1, \cdots, N\}, \quad (5)$$

where d_i is referred as the RBF scale which measures the smoothness of the function, and the tolerance parameter $\eta > 0$ is a problem dependent parameter and its optimal choice needs requires some subjective tuning.

Tukey's Boxplot Method

In statistics, the outliers are those data that fall far away from the median of the data set. We hypothesize that discontinuities or shocks are singular and rare energetic finite events in the solution of hyperbolic conservation laws and behave as the outliers. In this case, the outliers detection (OD) algorithm with Tukey's boxplot method can be employed here to increase the robustness of the shock detection method.



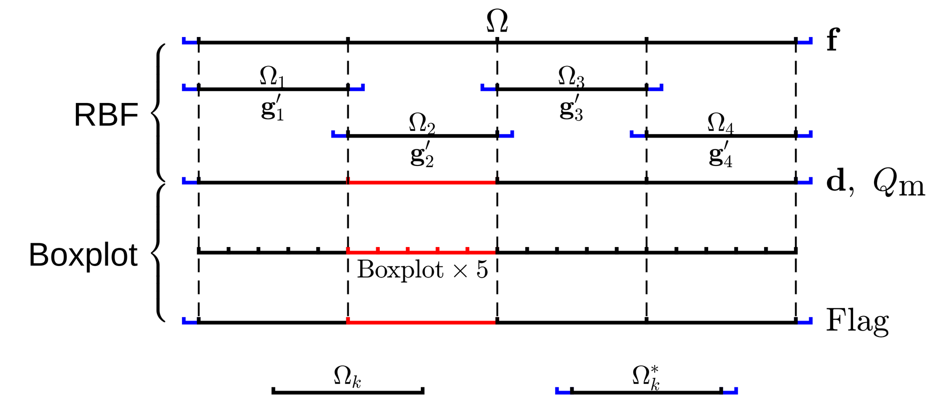
Where $\alpha = 1.5$ or 3 and Q_i are $i/4$ quartiles, $i = 1, 2, 3$. The outliers are out of the boxplot domain $\mathbf{\Omega}_f = [F_1, F_3]$.

The OD algorithm via Tukey's boxplot method

1. Define the first (third) quartile Q_1 (Q_3) which is the middle between the smallest (largest) value and the median of the data set $\mathbf{d}$ (by sorting the data set $\mathbf{d}$ in an ascending order).
2. Compute the global mean $Q_{\mathrm{m}} = \frac{1}{N+1}(\sum_{i=0}^{N} |d_i|)$.
3. Define the boxplot domain $\mathbf{\Omega}_f = [F_1, F_3]$ with the fences $F_1 = \min\{-Q_{\mathrm{m}}, Q_1 - 3\Delta Q\}$ and $F_3 = \max\{Q_{\mathrm{m}}, Q_3 + 3\Delta Q\}$.
4. Detect and flag the outliers by
$$\mathrm{Flag}_i = \begin{cases} 1, & d_i \notin \mathbf{\Omega}_f, \quad \text{(Outlier)}, \\ 0, & d_i \in \mathbf{\Omega}_f, \quad (\ \text{Inlier}). \end{cases} \quad (6)$$
5. Requires domain segmentation with data length $m \approx 20$ for a boxplot to be statistically meaningful.

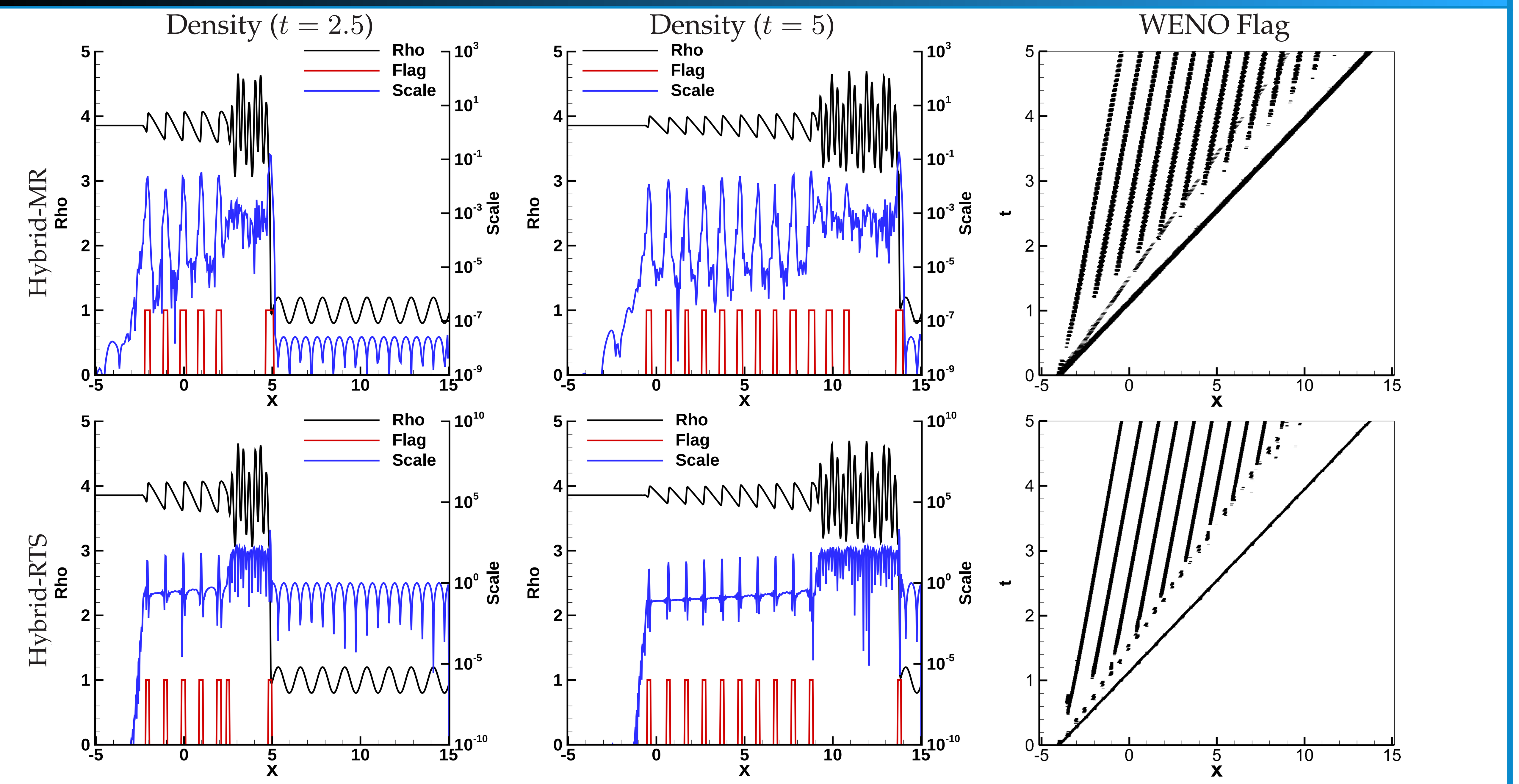
RBF Shock Detection Method

1. Uniformly divide the domain into N_S subdomains $\{\mathbf{\Omega}_k\}_{k=1}^{N_S}$ and overlapping subdomains $\{\mathbf{\Omega}_k^*\}_{k=1}^{N_S}$.
2. Approximate the derivative $\vec{g}'$ of solution at $\mathbf{\Omega}_k^*$ and the derivative at the physical domain is the unit of those at the subdomains $\{\mathbf{\Omega}_k\}_{k=1}^{N_S}$.
3. Compute RBF scales $\mathbf{d} = |\vec{g}'|^2$.
4. Detect the non-smooth stencils at the each subdomain $\mathbf{\Omega}_k^*$ by the OD algorithm with the mean Q_{m} being the global mean of $\mathbf{d}$ of the whole domain.

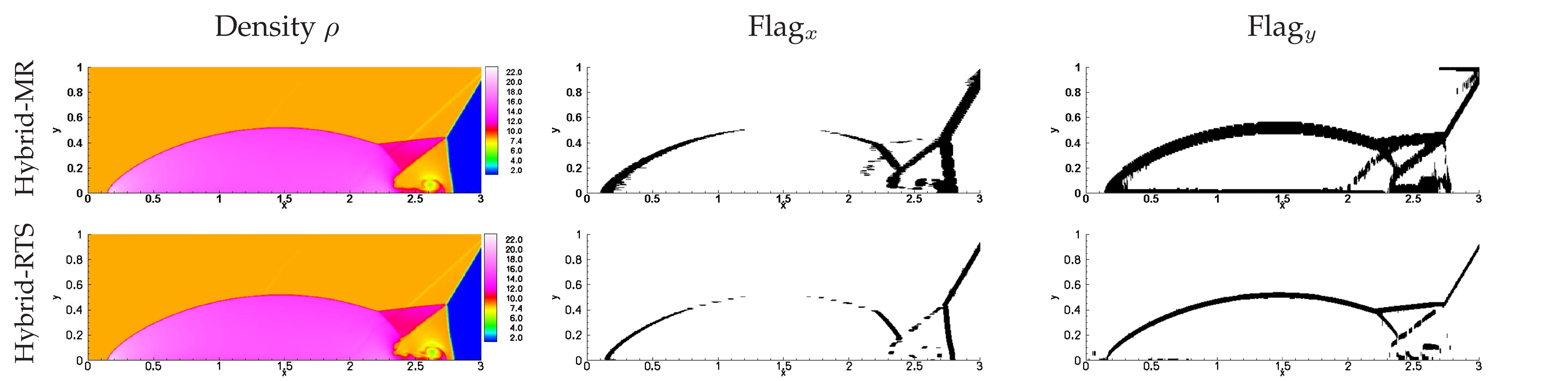


We refer to the hybrid Compact-WENO scheme with the RBF shock detection method as the Hybrid-RTS scheme.

Extended Mach 3 Shock-Density Problem ($N = 800$)



2D DMR Problem ($N \times M = 800 \times 200$)



Speedup Factors (SF)

Extended Mach 3 shock-density problem.

N	WENO-Z	Hybrid-MR		Hybrid-RTS	
	Time	Time	SF	Time	SF
800	2.2	1.7	1.3	1.3	1.8
1600	8.7	3.4	2.5	3.7	2.4
2400	19.2	5.8	3.3	7.8	2.5

2D DMR problem.

$N \times M$	WENO-Z	Hybrid-MR		Hybrid-RTS	
	Time	Time	SF	Time	SF
400×100	144	87	1.7	66	2.2
800×200	1240	539	2.3	430	2.9

Future Work

Research on the RBF shock detection method on unstructured mesh and its performance as a trouble-cell indicator in DG method.

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