Well-Balanced Nodal DG Methods for 1D Shallow Water Equation

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Introduction

An intelligent combination of the finite element and the finite volume methods, utilizing a space of basis and test functions that mimics the finite element method but satisfying the equation in a sense closer to the finite volume method, appears to offer many properties. This combination is exactly what leads to the discontinuous Galerkin finite element method (DG-FEM).

By carefully designing the numerical flux to reflect the underlying dynamics, the DG-FEM has more flexibility than in the classic FEM. Compared with the FVM, the DG-FEM overcomes the key limitation on achieving high-order accuracy on general grids by enabling this through the local element-based basis. This is all achieved while maintaining benefits such as local conservation and flexibility in the choice of the numerical flux.

In the nodal discontinuous Galerkin methods, we introduce local grid points, and express the polynomial through the associated interpolating Lagrange polynomial.

Shallow water equation is the hyperbolic systems of conservation laws with source terms(also called balance laws). These balance laws often admit steady state solutions in which the source term is exactly balanced by the flux gradients. Such cases, along with their perturbations, are very difficult to capture numerically. The objective of well-balanced schemes is to preserve exactly some of these steady state solutions.

Nodal DG Methods

Consider the nonlinear equation:

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = S, \; x \in [L, R] = \Omega, \qquad (1)$$

which is subject to the appropriate initial condition:

$$U(x, 0) = U_0(x).$$

We approximate Ω by K nonoverlapping elements, $x \in [x_l^k, x_r^k] = D^k$, and consider the following approximation space made up of polynomial functions: $V_h = \text{span}\,\{l_i(D^k)\}_{i=1}^N$.

We can obtain the strong form DG-FEM of (1):

$$\int_{D^k} (\partial_t U_h^k + \partial_x F_h^k - S_h^k) l_j^k \, dx = \int_{\partial D^k} \hat{\mathbf{n}} \cdot (F_h^k - F^*) l_j^k \, dx.$$

The choice of the numerical flux $F^* = F^*(U_h^-, U_h^+)$ is the Lax-Friedrichs flux:

$$F^{LF}(a, b) = \frac{F(a) + F(b)}{2} + \frac{C}{2}(b - a),$$

where $C \geq \max\limits_{\min(a,b) \leq s \leq \max(a,b)} |\frac{\partial F}{\partial U}(s)|$.

MUSCL Limiter

The classic MUSCL limiter is

$$\Pi^1 u_h^k(x) = \overline{u}_h^k + (x - x_0^k) m((u_h^k)_x, \frac{\overline{u}_h^{k+1} - \overline{u}_h^k}{h}, \frac{\overline{u}_h^k - \overline{u}_h^{k-1}}{h}),$$

where the TVB minmod function is

$$m(a_1, \cdots, a_l) = \begin{cases} a_1, & \text{if } |a_1| \leq Mh^2, \\ s \min\limits_{1 \leq i \leq l} |a_i|, & \text{if } |a_1| > Mh^2, \\ & \text{sign}_{1 \leq i \leq l}(a_i) = s, \\ 0, & \text{otherwise.} \end{cases}$$

1D shallow water equations

One-dimensional shallow water equation is:

$$\begin{cases} h_t + (hu)_x = 0, \\ (hu)_t + (hu^2 + \frac{1}{2} g h^2)_x = -g h b_x, \end{cases} \qquad (2)$$

where h denotes the water height, u is the velocity of the fluid, b represents the bottom topography and g is the gravitational constant.

Well-balanced DG scheme

A well-balanced DG scheme takes the form:

$$\int_{D^k} (\partial_t U_h^k + \partial_x F_h^k - S_h^k) l_j^k \, dx = \int_{\partial D^k} \hat{\mathbf{n}} \cdot (F_h^k - \widehat{F}) l_j^k \, dx.$$

We follow the hydrostatic reconstruction to set

$$h_j^{*,\pm} = \max(\,0,\; h_j^\pm + b_j^\pm - \max(b_j^+, b_j^-)\,),$$

and redefine the left and right values of U as:

$$U_j^{*,\pm} = \begin{pmatrix} h_j^{*,\pm} \\ h_j^{*,\pm} U_j^\pm \end{pmatrix}.$$

Then the well-balanced flux $\widehat{F}$ is given by

$$\widehat{F} = F^{LF} + \frac{g}{2} \begin{pmatrix} 0 \\ (h^-)^2 - (h^{*,-})^2 \end{pmatrix}.$$

We separate the source term integral into:

$$-\int_{D^k} g h b_x l_j^k \, dx = \int_{D^k} (\frac{g}{2}(b)^2)_x l_j^k \, dx + \int_{D^k} g(h + b) b_x l_j^k \, dx.$$

Combined with the above, we get the well-balanced DG-FEM of Shallow water equations.

In order to achieve the well-balanced property, we choose the limiter by two steps: the first one is to check whether any limiting is needed in a specific cell; and then we apply the MUSCL limiter on the variables in these cell.

Test for the exact C-property

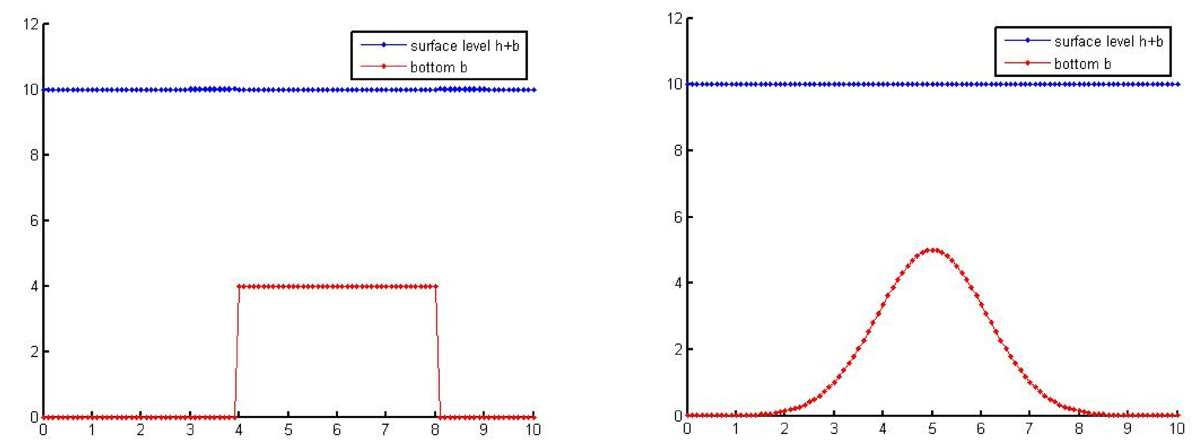

Figure 1: The surface level $h + b$ and the bottom b for the stationary flow.

Table 1: L^∞ error for different bottom topography

L^∞ error	h	hu
discontinuous bottom $b(x)$	3.0E-14	5.4E-13
smooth bottom $b(x)$	2.7E-15	1.9E-13

The two different functions for the bottom topography are:

$$b(x) = \begin{cases} 4, & \text{if } 4 \leq x \leq 8 \\ 0, & \text{otherwise} \end{cases}, \; x \in [0, 10], \qquad (3)$$

which is discontinuous, and

$$b(x) = 5 \exp(-\frac{2}{5}(x - 5)^2), \qquad (4)$$

which is smooth.

The initial data is the stationary solution:

$$h + b = 10, \quad hu = 0.$$

This steady state should be exactly preserved. We compute the solution until $t = 0.5$ using $N = 200$ uniform cells. The exact C-property is verified by the results plotted in Fig. 1. And the L^∞ errors for the water height h and the discharge hu are shown in Table 1.

The dam breaking problem over a rectangular bump

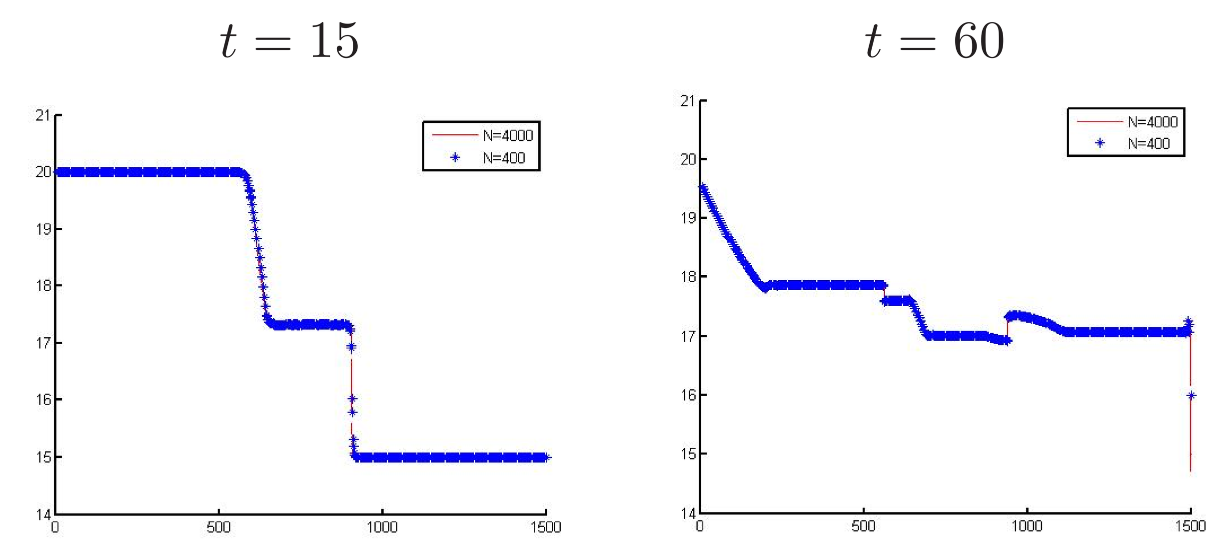

Figure 2: The numerical solution of surface level $h + b$ for the dam breaking problem.

We simulate the dam breaking problem over a rectangular bump, which takes the bottom topography:

$$b(x) = \begin{cases} 8, & \text{if } |x - 750| \leq 1500/8 \\ 0, & \text{otherwise} \end{cases}, x \in [0, 1500]. \qquad (5)$$

The initial conditions are

$$u(x, 0) = 0, \quad h(x, 0) = \begin{cases} 20 - b(x), & \text{if } x \leq 750, \\ 15 - b(x), & \text{otherwise.} \end{cases}$$

The numerical results with 400 uniform cells (and reference solution using 4000 uniform cells) are shown in Figs. 2, with time $t = 15$ and $t = 60$. The well-balanced DG-FEM scheme works well for this example, giving well resolved and non-oscillatory solutions.

Future Work

Well-balanced Nodal discontinuous Galerkin methods for computational oceanography and shallow water simulations with a realistic ocean floor topology.

References

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