Well-Balanced DG Methods for Shallow Water Equations

The Third Summer Workshop in Advanced Research in Applied Mathematics and Scientific Computing 2015, OUC, China

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Introduction

The shallow water equations with non-flat bottom topography play a significant role in the modeling and simulation of flows in rivers, lakes and coastal areas. Therefore, research on effective and accurate numerical methods for their solutions has attracted great attention in the past two decades.

The well-balanced discontinuous Galerkin methods for the shallow water equations can almost maintain the still water steady state exactly, and at the same time can almost preserve the nonnegativity of the water height without loss of mass conservation. Also the unstructured triangular meshes and the simple positivity-preserving limiter can be used in DG method. Some numerical examples show that the method can get good resolution for smooth and discontinuous solutions, verifying the positivity-preserving property, well-balanced property.

Shallow Water Equations

The two-dimensional shallow water equations take the form:

$$\begin{cases} h_t + (hu)_x + (hv)_x = 0, \\ (hu)_t + (hu^2 + \frac{1}{2}gh^2)_x + (huv)_y = -ghb_x, \\ (hv)_t + (huv)_x + (hv^2 + \frac{1}{2}gh^2)_x = -ghb_y, \end{cases}$$

where h denotes the water height, $(u,v)^T$ is the velocity vector, b represents the bottom topography and g is the gravitational constant.

For the easy presentation in next section, we denote the shallow water equations by

$$U_t + \nabla \cdot \mathbf{F}(U) = s(h,b),$$

where $U = (h, hu, hv)^T$, $\mathbf{F}(U) = (f(U), g(U))$ are the flux and $s(h,b)$ is the source term.

DG method

Let $\mathcal{T}_\tau$ be a family of partitions of the computational domain Ω. For each edge e_K^i $(i = 1,2,3)$ of K, we denote its outward unit normal vector by ν_K^i. Let $K(i)$ be the neighboring triangle along the edge e_K^i. We seek an approximation, still denoted by U with an abuse of notation, which belongs to the finite dimensional space

$$V_\tau = V_\tau^K \equiv \{\omega \in L^2(\Omega); \omega|_K \in P^k(K), \forall K \in \mathcal{T}_\tau\}.$$

We project the bottom function b into the same space V_τ, to obtain an approximation which is still denoted by b, again with an abuse of notation. Let $\mathbf{x}$ denote (x,y), then the numerical scheme is given by

$$\iint_K \partial_t U \omega d\mathbf{x} - \iint_K \mathbf{F}(U) \cdot \nabla \omega d\mathbf{x} + \sum_{i=1}^{3} \int_{e_K^i} \hat{\mathbf{F}}|_{e_K^i} \cdot \nu_K^i \omega ds = \iint_K s(h,b) \omega d\mathbf{x},$$

where $\omega(\mathbf{x})$ is a test function from the test space V_τ. The $\hat{\mathbf{F}}$ is numerical flux.

Well-Balanced DG Methods

The numerical flux $\hat{\mathbf{F}}$ in DG method is defined by

$$\hat{\mathbf{F}}|_{e_K^i} \cdot \nu_K^i = \mathcal{F}(U_i^{int(K)}, U_i^{ext(K)}, \nu_K^i),$$

where $U_i^{int(K)}$ and $U_i^{ext(K)}$ are the approximations to the values on the edge e_K^i obtained from the interior and the exterior of K.

In order to achieve the well-balanced property, we are interested in preserving the still water stationary solution exactly. The well-balanced numerical scheme takes the form:

$$\iint_K \partial_t U \omega d\mathbf{x} - \iint_K \mathbf{F}(U) \cdot \nabla \omega d\mathbf{x} + \sum_{i=1}^{3} \int_{e_K^i} \hat{\mathbf{F}}^*|_{e_K^i} \cdot \nu_K^i \omega ds = \iint_K s(h,b) \omega d\mathbf{x}.$$

The flux $\hat{\mathbf{F}}^*$ is computed based on the hydrostatic reconstruction technique. We set

$$h_i^{*,int} = \max\left(0,\ h_i^{int} + b_i^{int} - \max(b_i^{int}, b_i^{ext})\right),$$
$$h_i^{*,ext} = \max\left(0,\ h_i^{ext} + b_i^{ext} - \max(b_i^{int}, b_i^{ext})\right),$$

and redefine the value of $U_i^{*,int}$ as:

$$U_i^{*,int} = \begin{pmatrix} h_i^{*,int} \\ h_i^{*,int} u_i^{int} \\ h_i^{*,int} v_i^{int} \end{pmatrix},$$

the the definition of $U^{*,ext}$ is similar to $U^{*,int}$.

Then well-balanced flux $\hat{\mathbf{F}}^*$ is given by

$$\hat{\mathbf{F}}^*|_{e_K^i} \cdot \nu_K^i = \mathcal{F}(U_i^{int(K)}, U_i^{ext(K)}, \nu_K^i) + \langle \delta_{i,x}^*, \delta_{i,y}^* \rangle \cdot \nu_K^i$$

where

$$\delta_{i,x}^* = \left(0, \frac{g}{2}\left(h_i^{int(K)}\right)^2 - \frac{g}{2}\left(h_i^{*,int(K)}\right)^2, 0\right),$$
$$\delta_{i,y}^* = \left(0, 0, \frac{g}{2}\left(h_i^{int(K)}\right)^2 - \frac{g}{2}\left(h_i^{*,int(K)}\right)^2\right).$$

Positivity-Preserving Limiter

At time level n, given the water height DG polynomial $h_K(x)$ with its cell avergae $\bar{h}_K^n \geq 0$, to enforce the sufficient condition$h_K(x) \geq 0$, $\forall x \in S_k$, the $h_K(x)$ can be replaced by a linear scaling around the cell average:

$$\tilde{h}_K(\mathbf{x}) = \theta_K\left(h_K(\mathbf{x}) - \bar{h}_K^n\right) + \bar{h}_K^n,$$

where $\theta_K \in [0,1]$ is determined by

$$\theta_K = \min_{x \in S_K} \theta_x, \quad \theta_x = \min\left\{1, \frac{\bar{h}_K^n}{\bar{h}_K^n - h_K(\mathbf{x})}\right\}.$$

This limiter is conservative, positivity-preserving and high order accurate.

Notice the positivity-preserving limiter does not affect the well-balanced property.

Oblique hydraulic jump

Table 1: The comparison between exact solution and numerical solution

	β	$h\ (m)$	$\lvert u\rvert\ (m/s)$
exact solution	30°	1.5	7.956
numerical solution	$\approx 30^\circ$	1.498	7.954

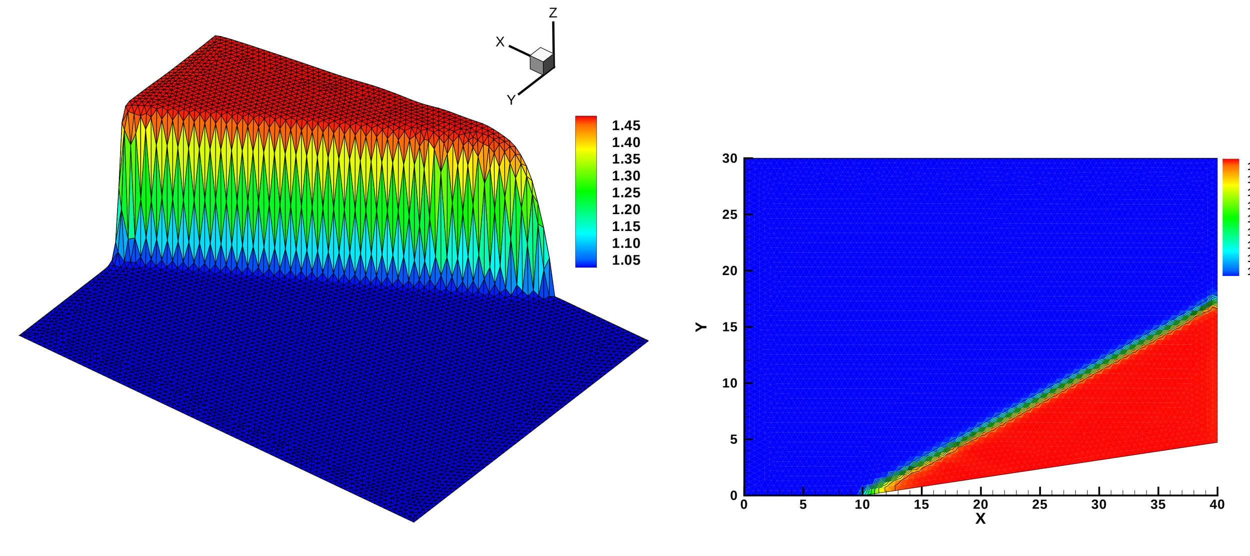

Figure 1: The surface level $h+b$.

In this example, an oblique hydraulic jump is induced by the means of an interaction between a supercritical flow and a converging wall deflected through an angle θ.

As the figure show, this is a $40m$ long river, the width of upstream is $30m$, and it is deflected at the point $(10,0)$ through the angle $\theta = 8.95^\circ$. The initial conditions are $h = 1m$, $u = 8.57m/s$, and $v = 0$.

The exact solution corresponding to the upstream flow and geometry imposed was calculated, the water depth in front of the wave is $h = 1.5m$, the velocity is $|u| = 7.9556m/s$ and the angle of the jump is $\beta = 30^\circ$. This example is often used to verify the correctness of the numerical algorithm.

The surface level $h+b$ is shown in Figure 1 by 3D and depth contour plot. From the Table 1, we can get that the numerical solution is basically identical to the exact solution.

Dam breaking problems

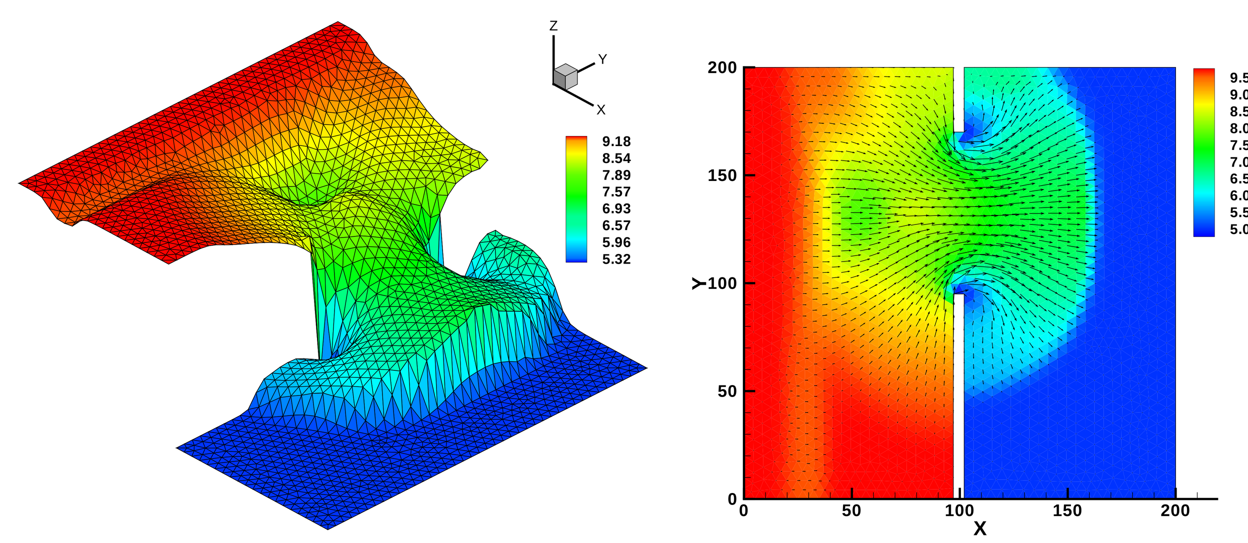

Figure 2: The surface level $h+b$ (right) and flow field (left).

To demonstrate the performance of the well-balanced DG method in two dimensional problems, we solve the two-dimensional dam-breaking problem with a flat bottom topography. The computational domain is $[0,200]^2$. The width of the dam is set to be $1:5$ and the breach is located at $x = 97$ to $x = 102$ and between $y = 65$ and $y = 135$. The boundary conditions are reflective, except that the inflow and outflow boundary conditions are transmissive. The final time is $t = 7.2$. The initial conditions are:

$$h(x,y,0) = \begin{cases} 10, & \text{if } x \leq 100 \\ 5, & \text{otherwise} \end{cases},$$
$$hu(x,y,0) = hv(x,y,0) = 0.$$

The water level and the flow field are shown in the Figure 2. The shock wave formed by the breaking of the dam is captured essentially oscillations free, and the shear vortical structures generated at the tips of the breach is well resolved.

Future Work

Well-balanced discontinuous Galerkin methods for:

- computational oceanography and shallow water simulations with a realistic ocean floor topology;
- numerical model of tidal bore in generalized estuary and in the Qiantang River.

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Acknowledgement

I would like to express my gratitude to all those who have helped me during the summer workshop. Frist, I gratefully acknowledge the finanical and academic support by Prof. Fang Qizhi, Prof. Gao Zhen, Prof. Xie, Shusen and Prof. Don Wai Sun. Second, I would like to thank Mr. Li Peng who help me a lot. Last my thanks would go to my partners.

Funding

- School of Mathematical Sciences, OUC.
- Startup grant by Ocean University of China (201412003).
- Natural Science Foundation of Shandong Province (ZR2012AQ003).
- National Natural Science Foundation of China (11201441).