Entropy Stable and Well-Balanced Discontinuous Galerkin Method for the Nonlinear Shallow Water Equations

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Abstract

The nonlinear shallow water equations (SWEs) are widely used to model the unsteady water flows in rivers and coastal areas, with extensive applications in ocean and hydraulic engineering.
The one-dimensional SWEs with a non-flat bottom is

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{Q})}{\partial x} = \mathbf{S}(b, \mathbf{Q}),$$

with the conservation variables, the flux and source terms

$$\mathbf{Q} = \begin{bmatrix} h \\ hu \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 0 \\ -ghb_x \end{bmatrix},$$

where $h(x,t)$ is the water height, $u(x,t)$ is the velocity, $m = hu$ is the momentum, $b(x,t)$ is the bottom topography, g is the gravitational constant.
In this work, we extend the idea in "Chan. J Comput Phys. 362:346-374, 2018" to the SWEs with a non-flat bottom topography, and establish an entropy stable scheme by adding additional dissipative terms. Careful approximation of the source term is included to ensure the well-balanced property. A simple positivity-preserving limiter, compatible with the entropy stable property, is included to guarantee the non-negative water heights.

Entropy solutions of PDEs

Consider systems of nonlinear conservation laws

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{Q})}{\partial x} = 0, \tag{0.1}$$

in a convex domain Ω.

- Entropy function is a convex function $U : \Omega \to R$ for (0.1) if there exist functions $F : \Omega \to R$, such that
$$U'(\mathbf{Q}) \frac{\partial \mathbf{f}}{\partial \mathbf{Q}} = \frac{\partial F(\mathbf{Q})}{\partial \mathbf{Q}}.$$
- Entropy variables $\mathbf{e}$ are
$$\mathbf{e} = U'(\mathbf{Q}).$$
- Entropy potential ψ, a scalar function in terms of the entropy function $\mathbf{e}$, is
$$\psi'(\mathbf{e}) = \mathbf{f}(\mathbf{Q}(\mathbf{e})), \psi(\mathbf{e}) = \mathbf{e}^T \mathbf{f}(\mathbf{Q}(\mathbf{e})) - F(\mathbf{Q}(\mathbf{e})).$$
- Conservation of entropy, when $\mathbf{Q}$ is smooth gives
$$\frac{\partial U(\mathbf{Q})}{\partial t} + \frac{\partial F(\mathbf{Q})}{\partial x} = 0.$$
- Entropy solution is one of weak solutions $\mathbf{Q}$ of (0.1), which for all entropy functions U,
$$\frac{\partial U(\mathbf{Q})}{\partial t} + \frac{\partial F(\mathbf{Q})}{\partial x} \leq 0.$$

Entropy conservative scheme

The entropy variables of the SWEs with the non-flat bottom are

$$\mathbf{e} = \begin{bmatrix} g(h+b) - \frac{1}{2}u^2 \\ u \end{bmatrix},$$

with the entropy U and the entropy potential ψ are

$$U = \frac{1}{2}hu^2 + \frac{1}{2}gh^2 + ghb, \qquad \psi = \frac{1}{2}gh^2 u.$$

Definition 1 *Let $W(x)$ be a bounded function on each element Ω^k. Define $D_h^x : H^1(\Omega_h) \to V_h$ as the operator which satisfies*

$$(D_h^x Q, W\omega)_\Omega = \sum_k \Big[\Big(\frac{\partial \mathbb{P}Q}{\partial x}, W\omega \Big)_{\Omega^k} + \frac{1}{2}\langle Q^+ - \mathbb{P}Q, W\omega n_x \rangle_{\partial\Omega^k} + \frac{1}{2}\langle \mathcal{E}(Q), \mathbb{P}(W\omega) n_x \rangle_{\partial\Omega^k} \Big],$$

for all $\omega \in V_h$, where n_x is the x-component of the outward normal vector.

This derivative operator D_h^x satisfies the global analogue of integration- by-parts property:

$$(D_h^x Q, W\omega)_\Omega = -(Q, D_h^x(W\omega))_\Omega + \langle Qn_x, W\omega \rangle_{\partial\Omega},$$

for $Q \in H^1(\Omega_h)$ and $W \in V_h$.

The weak derivative operator D_h^x is adopted to derive the entropy conservative DG methods for the SWEs, which take the form of

$$\Big(\frac{\partial \mathbf{Q}}{\partial t}, \omega \Big)_\Omega + \Big(\big(2D_h^x \mathbf{f}_S(\mathbf{Q}_e(x), \mathbf{Q}_e(y))\big)\big|_{y=x}, \omega \Big)_\Omega = \begin{pmatrix} 0 \\ (-gh_e D_h^x b_e, \omega)_\Omega \end{pmatrix}, \tag{0.2}$$

where

$$b_e = \mathbb{P}b, \quad h_e = \frac{1}{g}\Big(\mathbb{P}e_1 + \frac{1}{2}(\mathbb{P}e_2)^2 - g\mathbb{P}b \Big), \quad m_e = h_e \mathbb{P}e_2,$$

$$\mathbf{Q}_e = \mathbf{Q}(\mathbb{P}\mathbf{e}) = \begin{bmatrix} h_e \\ m_e \end{bmatrix}.$$

The entropy conservative fluxes for the one-dimensional SWEs with a non-flat bottom topography are:

Definition 2 *Let $\mathbf{f}_S(\mathbf{Q}_l, \mathbf{Q}_r)$ be a consistent and symmetric numerical flux. It is entropy conservative for SWEs if*

$$(\mathbf{e}_l - \mathbf{e}_r)^T \mathbf{f}_S(\mathbf{Q}_l, \mathbf{Q}_r) = (\psi_l - \psi_r) + \frac{1}{2}g(b_l - b_r)(m_l + m_r),$$

holds for the entropy variables $\mathbf{e}_j = \mathbf{e}(\mathbf{Q}_j)$ and the scaler entropy potential $\psi_j = \psi(\mathbf{e}(\mathbf{Q}_j))$, $j = l, r$.

Entropy conservative

Theorem 1 *For the global domain Ω, the DG methods (0.2) is globally entropy conservative in the sense that*

$$\Big(\frac{\partial U(\mathbf{Q})}{\partial t}, 1 \Big)_\Omega = \langle \psi, 1n_x \rangle_{\partial\Omega} - \langle \mathbf{f}(\mathbf{Q}_e), \mathbb{P}\mathbf{e}n_x \rangle_{\partial\Omega}.$$

Local conservation property

Theorem 2 *When the test function $\omega = 1$, the DG method (0.2) reduces to*

$$\Big(\frac{\partial h}{\partial t}, 1 \Big)_{\Omega_k} + \Big\langle f_S^{(1)}(\mathbf{Q}_e^+, \mathbf{Q}_e), n_x \Big\rangle_{\partial\Omega_k} = 0,$$

$$\Big(\frac{\partial m}{\partial t}, 1 \Big)_{\Omega_k} + \Big\langle f_S^{(2)}(\mathbf{Q}_e^+, \mathbf{Q}_e), n_x \Big\rangle_{\partial\Omega_k} = -\Big(gh_e \frac{\partial}{\partial x} b_e, 1 \Big)_{\Omega_k} - \Big\langle \frac{1}{2} gh_e [\![b_e]\!], n_x \Big\rangle_{\partial\Omega^k}.$$

When the bottom topography b is flat, it becomes

$$\Big(\frac{\partial \mathbf{Q}}{\partial t}, 1 \Big)_{\Omega^k} + \langle f_S(\mathbf{Q}_e^+, \mathbf{Q}_e), n_x \rangle_{\partial\Omega^k} = 0.$$

This means our method is locally conservative, similar as the conventional DG methods for the SWEs.

Entropy stable scheme

Following the idea of the Lax-Friedrichs numerical flux, we could add additional dissipative terms at the element interfaces, and construct the entropy stable DG methods in the form of

$$\Big(\frac{\partial \mathbf{Q}}{\partial t}, \omega \Big)_\Omega + \Big(\big(2D_h^x \mathbf{f}_S(\mathbf{Q}_e(x), \mathbf{Q}_e(y))\big)|_{y=x}, \omega \Big)_\Omega - \sum_k \frac{\alpha}{2} \Big\langle \begin{pmatrix} [\![h_e + b_e]\!] \\ [\![m_e]\!] \end{pmatrix}, \omega n_x \Big\rangle_{\partial\Omega^k} = \begin{pmatrix} 0 \\ (-gh_e D_h^x b_e, \omega)_\Omega \end{pmatrix} \tag{0.3}$$

where $\alpha = \max(|u| + \sqrt{gh})$ denotes the maximum eigenvalue of the Jacobian matrix of the flux.

Theorem 3 *The DG method (0.3) is semi-discretely entropy stable in the sense that*

$$\Big(\frac{\partial U(\mathbf{Q})}{\partial t}, 1 \Big)_\Omega \leq \langle \psi, 1n_x \rangle_{\partial\Omega} - \langle \mathbf{f}(\mathbf{Q}_e), \mathbb{P}\mathbf{e}n_x \rangle_{\partial\Omega}.$$

Well-balanced property

Theorem 4 *The entropy conservative method (0.2) and the entropy stable method (0.3) are both well-balanced for the still-water steady state solution, denoted by*

$u = 0$ and $h + b =$ constant C.

Positivity-preservation limiter

Theorem 5 *The entropy stable method (0.3) is compatible with the positivity-preserving limiter.*

Accuracy test

N	h L^1 error	h Order	hu L^1 error	hu Order
25	2.47E-3		2.09E-2	
50	1.16E-4	4.41	1.02E-3	4.35
100	1.23E-5	3.24	1.08E-4	3.24
200	1.02E-6	3.59	8.95E-6	3.60
400	1.11E-7	3.20	9.77E-7	3.20
800	1.30E-8	3.10	1.14E-7	3.10

Exact C-property test

Table 0.1: L^1 and L^∞ errors for the stationary solution with a discontinuous bottom.

N	L^1 error		L^∞ error	
	h	hu	h	hu
100	1.1E-13	5.4E-14	1.5E-13	3.7E-13
200	1.1E-13	5.0E-14	1.6E-13	3.2E-13
400	1.2E-13	4.0E-14	1.7E-13	2.6E-13

Dam-breaking problem

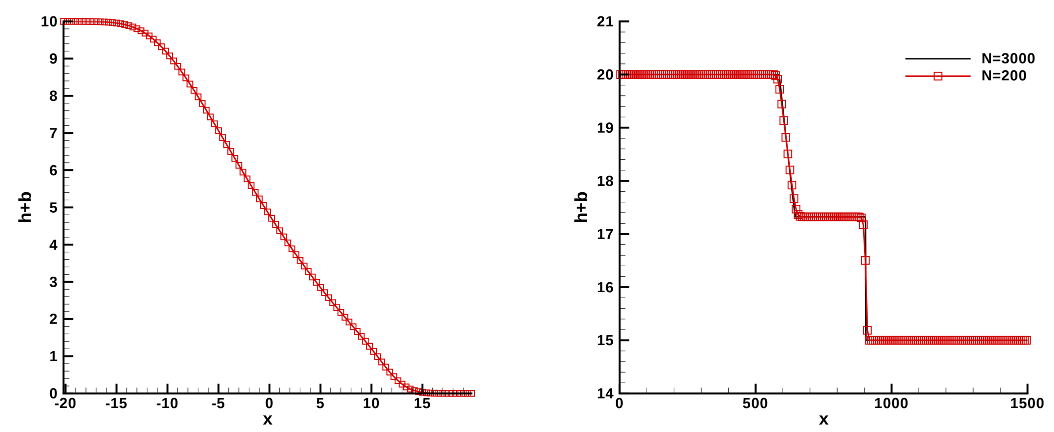

Figure 0.1: Dam-breaking problems with dry bed and wet bed.

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