One Application of MQ RBF Method: for detection of discontinuities

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Introduction

Radial basis function (RBF) methods have been widely used for various areas of application, such as machine learning, neural networks, image processing and reconstruction, solving partial differential equations (PDEs), and so on. RBF methods are referred to meshless methods because their centers can be distributed arbitrarily based on the given geometry, without any constraints from the method itself.

In this research, we mainly consider iterative adaptive multi-quadric (IAMQ) RBF methods for detection of discontinuities in 1D and 2D problems. For the 2D problems, we use a slice-by-slice approach which is simple to implement and computationally inexpensive.

MQ RBFs

As we all know, RBFs are defined using centers and a set of corresponding shape parameters. First, we consider a center set $\mathbf{X} = \{\mathbf{x}|\mathbf{x} \in \Omega \subset \mathbb{R}^d, d \geq 1\}$, and $\mathbf{x} = (x_1, x_2, \cdots, x_d)$.

Besides, we let $\mathbf{r}_j(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_j\|, \mathbf{x}_j \in \mathbf{X},\ \mathbf{x} \in \mathbf{X}$, which is Euclidean norm, where Ω is the given domain, ϵ_j is the shape parameter. Then, the different RBFs are defined by as follows:

Table 1: Some common choices of RBFs

Type of RBF	RBF $\Psi_j(\mathbf{x})$
Gaussian (GA)	$e^{(-\epsilon_j \mathbf{r}_j)^2}$
MQ	$\sqrt{\mathbf{r}_j^2 + \epsilon_j^2}$
Inverse MQ	$1/\sqrt{\mathbf{r}_j^2 + \epsilon_j^2}$
Inverse quadratic (IQ)	$1/(\mathbf{r}_j^2 + \epsilon_j^2)$

Besides, there are some other RBFs, but in this research, we choose MQ RBFs because they have exponential convergence and high precision in solving differential equations. The RBF interpolation of $f(\mathbf{x})$ is denoted by

$$\mathbf{s}(\mathbf{x}) = \sum_{j=1}^{N} \lambda_j \Psi_j(\mathbf{x}), \tag{1}$$

satisfying $\mathbf{s}(\mathbf{x}_j) = f(\mathbf{x}_j), j = 1, \cdots, N$, where λ_j is the expansion coefficient, N is the number of nodes.

$\Rightarrow \mathbf{M}\boldsymbol{\lambda} = \mathbf{f}$, where $\mathbf{M}_{i,j} = \Psi_j(\mathbf{x}_i)$.

Then the first derivative of $\mathbf{s}(\mathbf{x})$ is

$$\mathbf{s}'(\mathbf{x}) = \sum_{j=1}^{N} \lambda_j (\mathbf{x} - \mathbf{x}_j)/\Psi_j(\mathbf{x}), \quad \mathbf{x} \in \mathbf{X}, \tag{2}$$

$\Rightarrow \mathbf{D}\boldsymbol{\lambda} = \mathbf{s}'$, where $\mathbf{D}_{i,j} = (\mathbf{x}_i - \mathbf{x}_j)/\Psi_j(\mathbf{x}_i)$.

Remark: MQ RBF is a global RBF then has a high accuracy in interpolation, but there exists large amount of calculation in interpolation matrix computer.

1D IA method

In this work, we use iterative ϵ−adaptive method, and the main idea of ϵ−adaptive method is that ϵ_j vanish only at the centers where the local discontinuity exists or in the small neighborhood of the local discontinuity. In the process of iterative, we use the first order derivatives $\mathbf{s}'$ and $\boldsymbol{\lambda}$ to detect the local jump discontinuity.

1D Iterative Algorithm

Given: $\epsilon_i,\ \eta > 0,\ \delta > 0,\ \lambda_{old}$

Step 1: Compute $\boldsymbol{\lambda}$, $\mathbf{s}'$, $\mathbf{C} = |\mathbf{s}'||\boldsymbol{\lambda}|/\max(|\mathbf{s}'||\boldsymbol{\lambda}|)$

Step 2: Find set $\mathbf{S} = \{x_i | x_i \in X, \mathbf{C}(x_i) \geq \eta\}$

Step 3: Update ϵ, **let** $\epsilon_i = 0$ **at** $x_i \in \mathbf{S}$

Step 4: If $\|\lambda - \lambda_{old}\| > \delta$, **update** λ_{old}, **repeat Step** $1 \rightarrow 3$.

Numerical examples:

In this test, we give the results of Multiple jump discontinuities and ρ from the 1D shock-wave interaction equations using the WENO-Z method at $t = 2$ presented Fig.1. And we choose $\epsilon_0 = 0.1, \eta = 0.5, \delta = 1.0e-7$.

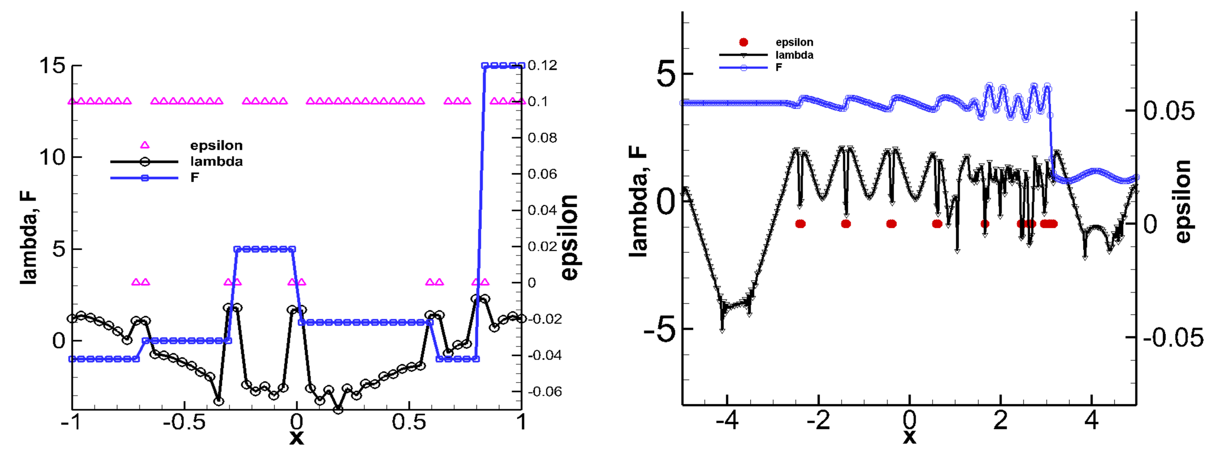

Figure 1: The discontinuous function $\mathbf{f}$, $\boldsymbol{\epsilon}(\epsilon_j = 0)$ and the $|\boldsymbol{\lambda}|$ in logarithmic scale with $N = 50$ (Left) and $N = 300$ (Right).

Seeing the results, we know that the method is very reliable, however, the biggest disadvantage of the method is costing more time.

References

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[2] J.-H. Jung, V. Durante, An iteratively adaptive multiquadric radial basis function method for the detection of local jump discontinuities, Appl. Numer. Math. 59 (2009) 1449-1466.

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2D method and tests

Because the global method is expensive, we use slice-by-slice approach for 2D problems. Further more, this method is very suitable for parallel computation and domain decomposition, that can promote the computational efficiency more.

Numerical examples:

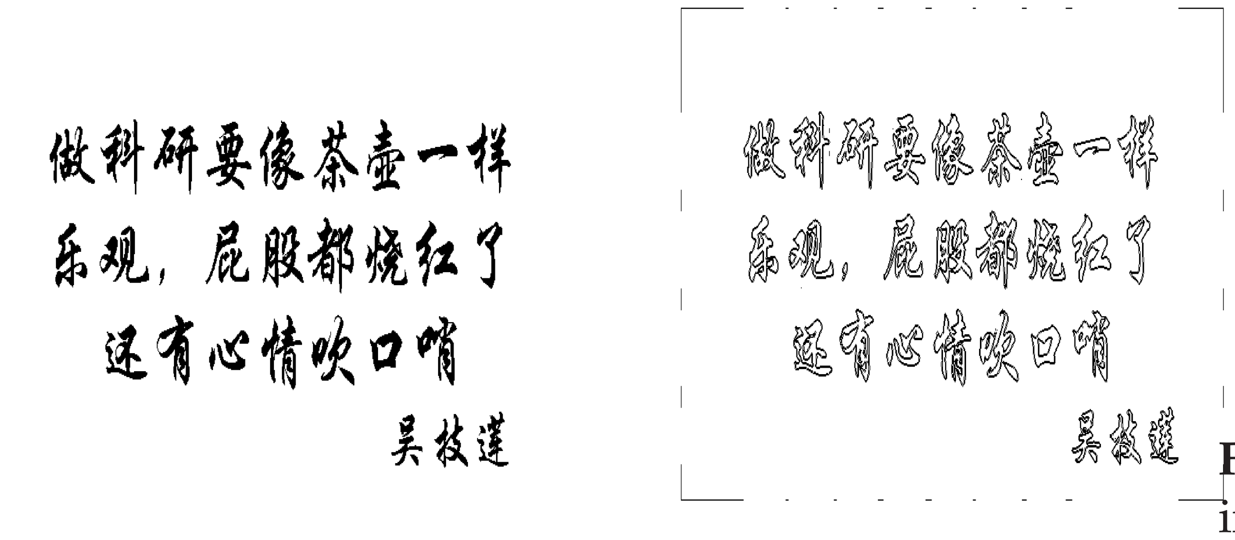

Figure 2: Left: Chinese characters map. Right: Edge map of the image

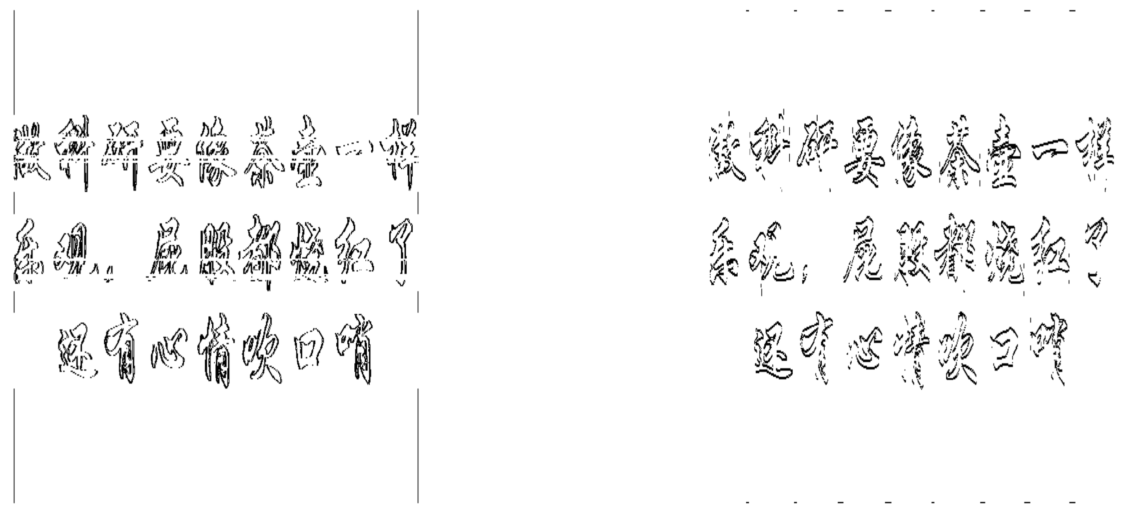

Figure 3: The x-slice edges detected (left), y-slice edges detected (right).

For Figs. 2-4, the size of images is 409×582, and the CPU times are 211.91s, 105.30s(left) and 83.63s, 166.97s, respectively.

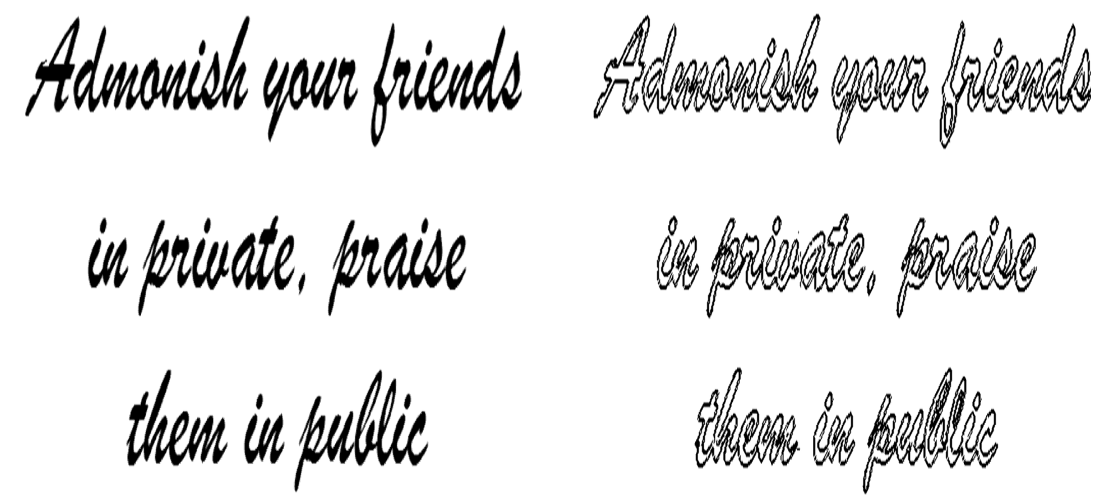

Figure 4: Left: English characters. Right: Edge map of the image

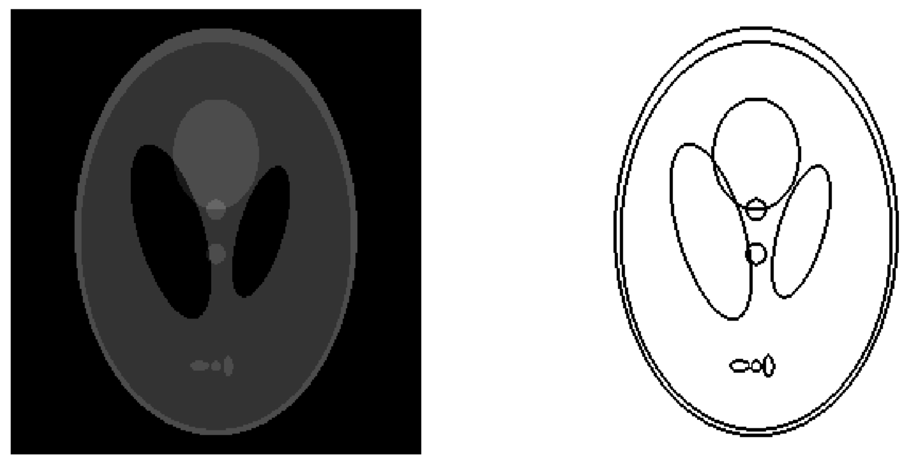

Figure 5: Shepp-Logan phantom map on 256×256 array. Left: The original square image. Right: The edge map detected at the last iteration step.

Future Work

- We plan to further study the RBFs and use parallel algorithm or domain decomposition technique to improve the efficiency.
- We shall use other techniques to improve the algorithm and to solve 3D problems.
- We will consider solving PDE problems.

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