WENO/ENO based Closest Point Method for Surface Hyperbolic Equation

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Introduction

The closest point method is an embedding method which uses a closest point representation of the surface to solve PDEs on surfaces. In the classical formulation, the discretization is carried out in a neighborhood of the surface using standard finite difference schemes and barycentric Lagrangian interpolation. The extension via the closest point mapping is also used as part of the development for other methods for solving surface PDEs.

In this paper, we use WENO finite difference scheme and ENO interpolation scheme to replace the linear difference scheme and linear interpolation scheme in order to solve discontinuity problem. And hybrid method can combine linear scheme with nonlinear scheme, which can combine the advantages of the two schemes. Nonlinear scheme can solve discontinuity but has large dissipation and dispersion and cost a lot of time. Linear scheme can't solve discontinuity but has small dissipation and dispersion and can be computed fast.

Closest point method

The classical closest point method is a simple numerical method for approximating the solution of PDEs on surfaces. In this section, we review the method and its components.

To begin, the closest point to the surface is defined:

Let $\Gamma \subset R^d$ be a surface, Ω be a narrow tube surrounding the surface $\Omega \supset \Gamma$ and $\mathbf{z}$ be the points on the Ω. Then,

$$\mathbf{x} = \mathrm{cp}_\Gamma(\mathbf{z}) = \arg\min D(\mathbf{x}, \mathbf{z})$$

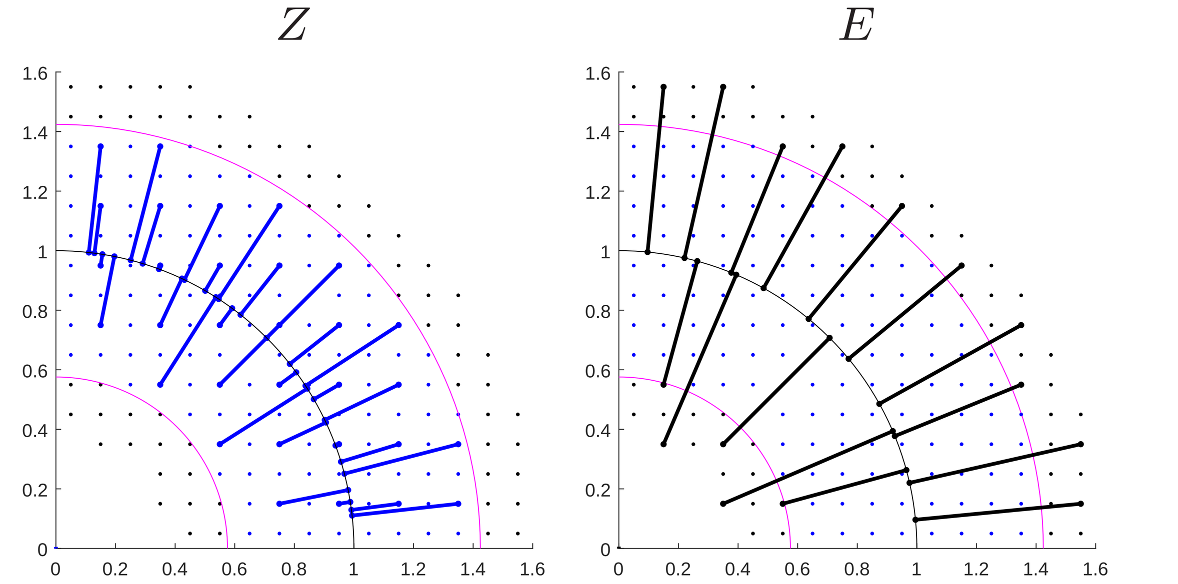
is the closest point of $\mathbf{z}$ to the surface Γ. $D(\mathbf{x}, \mathbf{z}) = \|\mathbf{x} - \mathbf{z}\|_2$ is the distance between $\mathbf{z}_j$ and $\mathbf{x}_j$.

Evolution of the embedding equation may be carried out in a similar fashion, to yield the explicit closest point method. Specifically, given a closest point representation of a surface Γ, the explicit closest point method alternates between the following two steps:

1.**Closest point extension.** Carry out a constant-along-normal extension of $u : \Gamma \to R$ to yield $\tilde{u} : \Omega_1 \to R$ by $\tilde{u}(z) = u(cp_\Gamma(z))$, for each z in the tubular computational domain $\Omega_1 \supset \Gamma$ and the width of the tubular is R_Z. Similarly, carry out a constant-along-normal extension of $u : \Gamma \to R$ to yield $\tilde{u} : \Omega_2 \to R$ by $\tilde{u}(e) = u(cp_\Gamma(e))$, for each e in the larger tubular computational domain $\Omega_2 \supset \Gamma$ and the width of the tubular is R_E. $R_E > R_Z$, so $\Omega_2 \supset \Omega_1$. In our method, Ω_1 is the interpolation domain and Ω_2 is the difference domain. R_E and R_Z in depended on the order of scheme.

2.**Evolution**. The PDE is solved on the difference domain Ω_2 in the embedding space for one time step (or one stage of a Runge-Kutta method). Interpolate the points $cp_\Gamma(e) \subset \Gamma$ by the interpolation domain Ω_1, then update the points E by closest point extension.

Closest point method



Algorithm

Given a collection of Cartesian grid points $Z = \{\mathbf{z}_j\}_{j=1}^{n_Z}$ in a small tubular domain Ω and a collection of Cartesian grid points $E = \{\mathbf{e}_k\}_{k=1}^{M_e}$ in a large tubular domain, which contain the surface Γ, the algorithm of WENO method for a time dependent PDE consists of the following steps:

1. Compute the set of surface points $\mathbf{X} = \{\mathbf{x}_j\}_{j=1}^{n_Z} \in \Gamma$ via the closest point representation of the surface Γ:$\mathbf{x}_j = cp_\Gamma(\mathbf{z}_j)$, for $\mathbf{z}_j \in Z, j = 1, \ldots, n_Z$.
2. Detect the point $\mathbf{Z}$ in x direction and y direction by RBF-FD shock detection to find the discontinuity point.
3. Compute the time dependent PDE by 3th Runge-Kutta scheme, for each stage:
   (a) Based on the result of shock detection, compute the derivatives of discontinuity points by WENO scheme and compute the derivatives of smooth points by finite difference scheme in two direction, so that get the gradient of point $\mathbf{Z}$.
   (b) Compute the gradient of point $\mathbf{X}$ by the relationship with $\mathbf{Z}$.
   (c) Perform a Runge-Kutta time-step and closest point extension for each point $\mathbf{z}$.
   (d) Compute the approximate value $\mathbf{x}_k = cp_\Gamma(\mathbf{e}_k)$ on Γ. If there is any discontinuity point in interpolation stencil, using ENO interpolation; If there is no discontinuity point in interpolation stencil, using Lagrange interpolation.
4. Compute equation by step 1-3 until the finial time.
5. Computed the value of uniform distribution point on the circle by interpolation via the point on Γ.

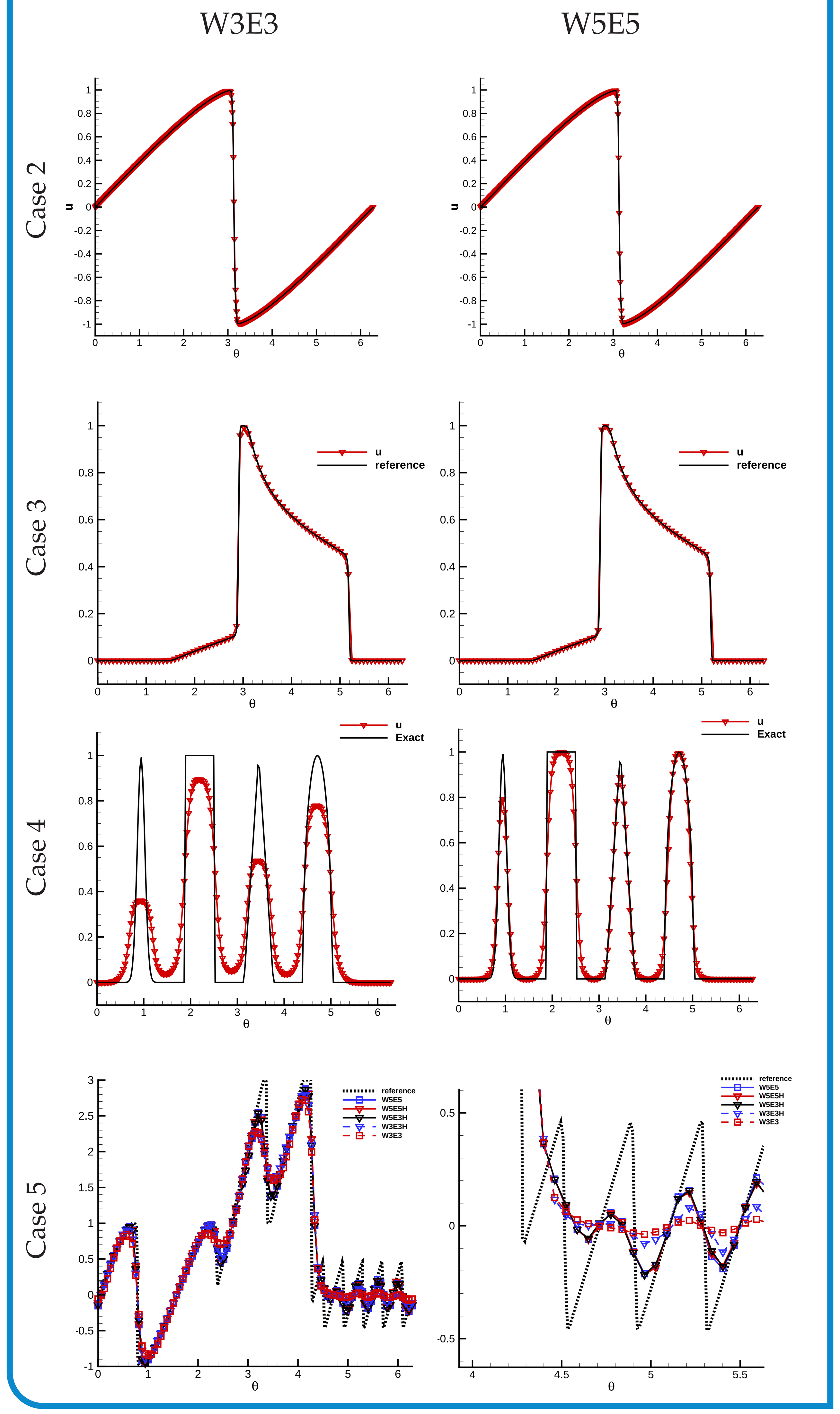
In the following, we show the accuracy tests by the wave equation (Case 1) with sine wave and some numerical example with discontinuities (such as burgers equation [1] (Case 2), Buckley-Leverett problem [2] (Case 3), unsmooth wave equation [2] (Case 4) and Compound wave [4] (Case 5)).

Accuracy tests

Consider the Case 1 problem with $\Delta t = \Delta x^{R/3} = \Delta y^{R/3}$, where R is the order of WENO and ENO.

Scheme	N^2	N_z	N_e	Error	Order
W3E3	40^2	528	248	1.2E-3	—
	80^2	1056	500	7.9E-5	3.92
	160^2	2112	1028	9.6E-6	3.04
	320^2	4256	1972	1.1E-6	3.11
	640^2	8520	3988	1.4E-7	2.97
W5E5	40^2	880	352	3.3E-3	—
	80^2	1776	752	2.5E-5	7.04
	160^2	3544	1504	2.1E-7	6.40
	320^2	7104	3020	3.3E-9	5.99
	640^2	14216	6024	6.2E-11	5.76

Numerical examples



CPU time

Case	N^2	W5E5	Hybrid-W5E5		Hybrid-W5E3	
		Time	Time	SF	Time	SF
1	80^2	71.7	10.0	7.2	7.25	9.8
	160^2	266.1	36.9	7.2	27.5	9.6
	320^2	1132.3	153.8	7.3	108.4	10.4
2	80^2	104.9	27.4	3.8	14.6	7.2
	160^2	422.0	94.8	4.5	54.7	7.7
	320^2	1767.1	358.6	4.9	205.4	8.6
3	80^2	195.3	67.9	2.9	31.5	6.2
	160^2	779.5	193.5	4.0	105.2	7.4
	320^2	3233.6	607.0	5.3	364.4	8.9
4	80^2	424.8	310.4	1.4	116.2	3.6
	160^2	1695.6	965.4	1.8	372.5	4.5
	320^2	6778.3	2616.2	2.6	1132.5	5.9
5	80^2	66.1	46.7	1.4	18.0	3.6
	160^2	262.8	142.9	1.8	54.8	4.8
	320^2	1036.1	368.3	2.8	166.2	6.2

Conclusion

In summary, WENO scheme and ENO scheme can solve discontinuity problem, there is no oscillation in shock. In low resolution, less dissipative grid points are solved via the high order WENO and ENO scheme than the low order WENO and ENO scheme. For the same order scheme, the hybrid scheme is better than standard scheme, because hybrid scheme handle smooth part better than standard scheme. And the result of 5th order WENO and 5th order ENO scheme closely approximates the result of hybrid scheme 5th order WENO and 5th order ENO scheme.

Future Work

In the future, we will extend the closest point method with WENO and ENO to hyperbolic conservation laws and higher dimensional surface.

References

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