Non-oscillatory Central Differencing for Hyperbolic Conservation Laws

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Introduction

Many of the recently developed high-resolution schemes for hyperbolic conservation laws are based on upwind differencing. The building block of these schemes is the averaging of an approximate Godunov solver; its time consuming part involves the field-by-field decomposition which is required in order to identify the "direction of the wind".

Instead, use a building block the more robust Lax-Friedrichs (LxF) solver.

- The main advantage: No Riemann problems are solved and hence field-by-field decompositions are avoided.
- The main disadvantage: The excessive numerical viscosity typical to the LxF solver.

Compensate for it by using high-resolution MUSCL-type interpolants. Numerical experiments show that the quality of the results obtained by such convenient central differencing is comparable with those of the upwind schemes.

Central Differencing Method

High-resolution schemes, which approximate the one dimensional system of conservation laws

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0, \tag{1}$$

are based on upwind differencing $\bar{v}(x,t) = v_j(t), x_{j-\frac{1}{2}} \leq x \leq x_{j+\frac{1}{2}}$.

To proceed in time, it can be expressed in terms of the Riemann solver, $R(\frac{x}{t}, w_l, w_r)$,

$$v(x, t+\triangle t) = R(\frac{x - x_{j+\frac{1}{2}}}{\triangle t}; v_j(t), v_{j+1}(t)), x_j \leq x \leq x_{j+1}.$$

This solution is then projected back into the space of piecewise constant gridfunctions. The following integral is used to get the average value to satisfy the constant in the whole section.

$$v_j(t+\triangle t) \equiv \bar{v}(x, t+\triangle t) = \frac{1}{\triangle x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} v(y, t+\triangle t)dy.$$

Integration over a typical cell $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [t, t+\triangle t]$ yields

$$v_j(t+\triangle t) = v_j(t) + \lambda[f(R_{j-\frac{1}{2}}) - f(R_{j+\frac{1}{2}})],$$

where $R_{j\pm\frac{1}{2}} = R(0^{\pm}; v_{j\pm 1}(t), v_j(t)), \lambda \equiv \frac{\triangle t}{\triangle x}$.

This shows the upwind property of the Godunov scheme.

Upwind Scheme

Consider a variable coefficient equation

$$\frac{\partial u}{\partial t} + a(x)\frac{\partial u}{\partial x} = 0.$$

At this point, because $a(x)$ may change the sign, among $a_j = a(x_j)$, the corresponding format is

$$\frac{u_j^{n+1} - u_j^n}{\triangle t} + a_j\frac{u_j^n - u_{j-1}^n}{\triangle x} = 0, a_j \geq 0, \tag{2}$$

$$\frac{u_j^{n+1} - u_j^n}{\triangle t} + a_j\frac{u_{j+1}^n - u_j^n}{\triangle x} = 0, a_j < 0, \tag{3}$$

Lax-Friedrichs Scheme

A high resolution approximation of (1), which is based on the staggered form of the Lax-Friedrichs (LxF) scheme,

$$v_{j+\frac{1}{2}}(t+\triangle t) = \frac{1}{2}(v_j + v_{j+1}) - \lambda[f(v_{j+1}(t)) - f(v_j(t))],$$

and the non-staggered version of the LxF scheme,

$$v_j(t+\triangle t) = \frac{1}{4}[v_{j+1}(t) + v_{j-1}(t)] - \frac{\lambda}{2}[f(v_{j+1}(t)) - f(v_{j-1}(t))]$$

The Modified LxF Scheme

In order to ensure that these schemes are also non-oscillatory, the numerical derivatives, $\frac{1}{\triangle x}w_j'$, should satisfy for every gridfunction $w = \{w_j\}, \triangle w_{j+\frac{1}{2}} = w_{j+1} - w_j$,

$$0 \leq w_j' \cdot sgn(\triangle w_{j\pm\frac{1}{2}}) \leq C \cdot |MinMod\{\triangle w_{j+\frac{1}{2}}, \triangle w_{j-\frac{1}{2}}\}|,$$

$$MinMod\{x, y\} = \frac{1}{2}[sgn(x) + sgn(y)] \cdot \min(|x|, |y|).$$

Now, a possible characteristic-wise choice for the numerical derivatives, $v_{j,k}' = \hat{R}_{j,k}MinMod\{\hat{\alpha}_{j+\frac{1}{2},k}, \hat{\alpha}_{j-\frac{1}{2},k}\}$, and the numerical flux derivatives can be calculated as $f_j' = \hat{A}_{j+\frac{1}{2}}v_j'$, where $\hat{A}_{j+\frac{1}{2}}$ is a Roe Matrix, $\hat{A}_{j+\frac{1}{2}} = A(v_j, v_{j+1})$,

$$f_{j,k}' = \hat{R}_{j,k}MinMod\{\hat{\alpha}_{j+\frac{1}{2},k}\hat{c}_{j+\frac{1}{2},k}, \hat{\alpha}_{j-\frac{1}{2},k}\hat{c}_{j-\frac{1}{2},k}\}.$$

The central differencing schemes can be rewritten

$$v_{j+\frac{1}{2}}(t+\triangle t) = \frac{1}{2}[v_j(t) + v_{j+1}(t)] - \lambda[g_{j+1} - g_j],$$

where the so-called modified numerical flux, is given by

$$g_j = f(v_j(t+\frac{\triangle t}{2})) + \frac{1}{8\lambda}v_j', \quad v_j(t+\frac{\triangle t}{2}) = v_j(t) - \frac{1}{2}\lambda f_j'.$$

The Euler Equation

The Euler equation can be written as

$$\frac{\partial Q}{\partial t} + \frac{\partial F(Q)}{\partial x} = 0, \tag{4}$$

where

$$\mathbf{Q} = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \quad \mathbf{F}(\mathbf{Q}) = \begin{bmatrix} m \\ \rho u^2 + p \\ u(E+p) \end{bmatrix}, \tag{5}$$

Here $\rho, u, m = \rho u$, p and E is respectively the density, velocity, momentum, pressure and total energy. p is the equation of stage.

The Roe matrix, $\hat{A}_{j+\frac{1}{2}}$, is associated with the eigensystem $\{\hat{c}_{j+\frac{1}{2},k}, \hat{R}_{j+\frac{1}{2},k}\}$, where $\hat{c}_{j+\frac{1}{2},k}$ is the eigenvalues.

Systems of Conservation Laws

When describe how to extend our scalar family of central differencing schemes to the one dimensional system of conservation laws,

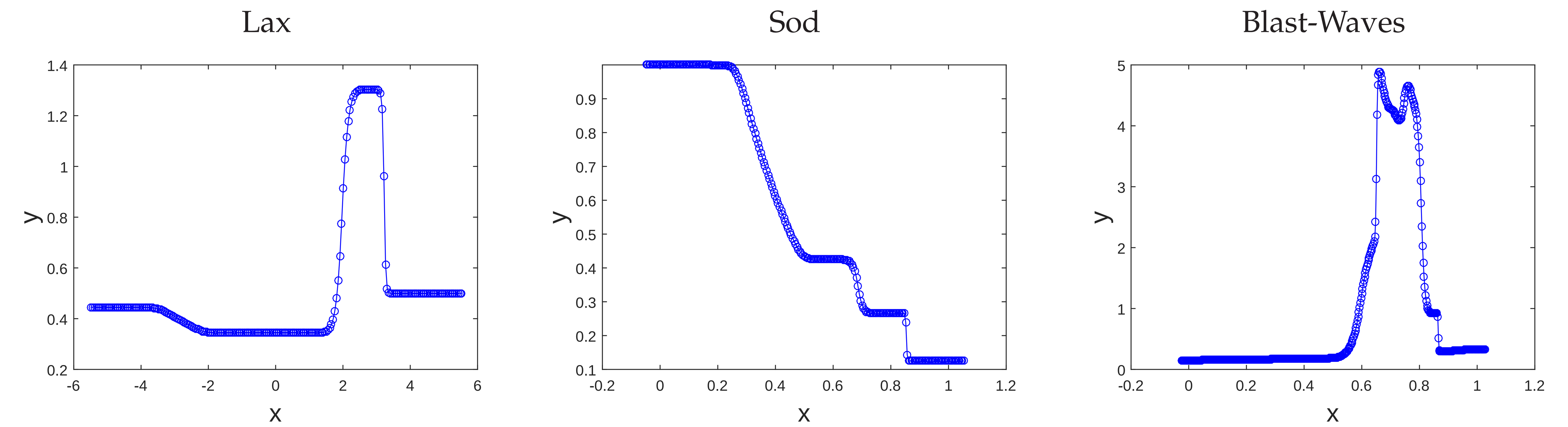
$$\frac{\partial u}{\partial t} + \frac{\partial (f(u))}{\partial x} = 0.$$

Here $u(x,t)$ is the unknown N-vector of the form and f(u) is the flux vector: $u = (u_1(x,t), u_2(x,t), ..., u_N(x,t))^T$, $f(u) = (f_1(u), f_2(u), ..., f_N(u))^T$,

The second contact field associated with $\hat{R}_{j+\frac{1}{2},2}$ is independent of the square root which is required only in the computation of the mean value sound speed $\hat{c}_{j+\frac{1}{2}}$.

So isolate the less expensive (i.e., square root-free) characteristic projection on the contact field, and use the component-wise approach for the other two fields.

Numerical Results for 1D Euler Equation



Colusion: The modified 2 order scheme is close to the exact solution.

Results for 1D Wave Equation

Table 1: Accuracy on 1D wave equation, where the exact solution is $f = \sin(\pi(x-t))$.

N	L_2 error	Order
80	1.4e-2	1.89
160	3.8e-2	1.90
320	9.7e-3	1.99
640	2.3e-3	2.07

Future Work

- Present numerical examples which demonstrate the performance of high resolution central differencing schemes for systems of conservation laws.
- Solve the 3 order nonoscillatory central scheme.
- Solve the high accurate finite differences based on RBF-FD problem.

References

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