New Modified Fifth Order WENO-Z Scheme

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1. Introduction

In the reconstruction step of $(2r-1)$ order weighted essentially non-oscillatory conservative finite difference schemes (WENO) for solving hyperbolic conservation laws, nonlinear weights α_k and ω_k, such as the WENO-JS weights by Jiang et al. and the WENO-Z weights by Borges et al., are designed to recover the formal $(2r-1)$ order (optimal order) of the upwinded central finite difference scheme when the solution is sufficiently smooth. It was recently shown that, for any design order $(2r-1)$, ϵ should be of $\Omega(\Delta x^2)$ ($\Omega(\Delta x^m)$ means that $\epsilon \geq C\Delta x^m$ for some C independent of Δx, as $\Delta x \to 0$) for the WENO-JS scheme to achieve the optimal order, regardless of critical points. The optimal order of the WENO-Z scheme can be guaranteed with a much weaker condition $\epsilon = \Omega(\Delta x^m)$, where $m(r,p) \geq 2$ is the optimal sensitivity order, regardless of critical points. But a smaller ϵ allows a better essentially non-oscillatory shock capturing as it does not over-dominate over the size of β_k. In this paper, we design a new modified fifth order WENO-Z scheme which can guarantee the optimal order of accuracy at presence of critical points with a smaller ϵ.We will give proof of our theory and numerically results on smooth function in the presence of critical points to confirm our theory.

2. The WENO-Z Scheme

The novel idea of the WENO-Z scheme (WENO-Z) [1, 2] is the inclusion of higher order information obtained from a global optimal order smoothness indicator τ_r in the formation of the nonlinear weights. This new global optimal order smoothness indicator τ_r is built using cell-averaged values in the whole S^{2r-1} stencil in the form of a linear combination of β_k, that is,

$$\tau_r = \left|\sum_{k=0}^{r-1} c_k\beta_k\right|,$$

where c_k are given constants (see [2]). For example, the global optimal order smoothness indicator τ_5 of the fifth order WENO-Z scheme is

$$\tau_5 = |\beta_0 - \beta_2|.$$

Its leading truncation error has been shown to be $O(\Delta x^5)$. In the $(2r-1)$ order WENO-Z scheme, the general definitions of the unnormalized and normalized nonlinear weights α_k^z and ω_k^z, respectively, are defined by

$$\alpha_k^z = d_k\left(1+\left(\frac{\tau_r}{\beta_k+\epsilon}\right)^p\right),\ \omega_k^z = \frac{\alpha_k^z}{\sum_{j=0}^{r-1}\alpha_j^z},\ k=0,\ldots,r-1.$$

As with the WENO-JS weights, p and ϵ are the power and sensitivity parameters, respectively.

3. The New WENO-Z Scheme

In this section, we developed a new modified fifth order WENO-Z scheme call WENO-Z$_{cp}$ scheme which can get the optimal order at presence critical points and with a smaller ϵ.

In the $(2r-1)$ order WENO-Z$_{cp}$ scheme, the idea is to add a new term to the WENO-Z weights, resulting in the formula

$$\alpha_k^z = d_k\left(1+\Phi\left(\frac{\tau_r}{\beta_k+\epsilon}\right)^p\right),$$

$$\omega_k^z = \frac{\alpha_k^z}{\sum_{j=0}^{r-1}\alpha_j^z},\quad k=0,\ldots,r-1$$

where Φ is

$$\Phi = min\{1,\phi\},\quad \phi = \sqrt{(|\beta_0-2\beta_1+\beta_2|)}$$

and it's Taylor series expansions at x_i are

$$\begin{aligned}\phi^2 =| &-\left(a_{14}f_i^{(1)}f_i^{(3)}\right)\Delta x^4\\ &-\left(a_{16}f_i^{(1)}f_i^{(5)} - a_{26}f_i^{(2)}f_i^{(4)} - a_{36}(f_i^{(3)})^2\right)\Delta x^6\\ &-\left(a_{18}f_i^{(1)}f_i^{(7)} - a_{28}f_i^{(2)}f_i^{(6)}\right)\Delta x^8\\ &-\left(-a_{38}f_i^{(3)}f_i^{(5)} - a_{48}(f_i^{(4)})^2\right)\Delta x^8\\ &+\left(a_{410}f_i^{(4)}f_i^{(6)} - a_{510}(f_i^{(5)})^2\right)\Delta x^{10} + O(\Delta x^{11})\ |\end{aligned}$$

4.1. Analysis of Accuracy

let

$$\gamma = \phi\left(\frac{\tau_{2r-1}}{\beta_k+\epsilon}\right)^p \tag{1}$$

according to [5], for optimal order, γ needs to satisfied the relation

$$\theta(\gamma) \leq r-1 \tag{2}$$

assuming that $\theta(\epsilon) \leq \theta(\beta)$, that means

$$\theta(\epsilon) \leq \theta(\tau_{2r-1}) + \frac{\theta(\phi)-(r-1)}{p}$$

In Table 1, the integer parts of the optimal sensitivity order $\theta(\epsilon)$ for WENO order 5 , $n_{cp} = 0,1,2,3$ and $p = 1,2,3$ are given.

4.1. Analysis of Accuracy

Table 1: The optimal sensitivity order $\theta(\epsilon)$ for order 5 .

			$\epsilon = \Omega(\Delta x^m)$		
n_{cp}	$\theta(\tau_5)$	$\theta(\phi)$	$p=1$	$p=2$	$p=3$
0	5	2	5	5	5
1	5	3	6	5	5
2	7	3	8	7	7
3	9	4	11	10	9

4.2. Convergence At Critical Points

Table 2: Rate of Convergence At Second Order Critical Points for $P=2, \epsilon = \Delta x^m$.

N	WENO-Z		WENO-Z$_{cp}$	
	Error	Order	Error	Order
80	0.47E-04	5.01	0.65E-04	5.40
160	0.15E-05	4.96	0.55E-06	6.88
320	0.20E-07	6.24	0.42E-08	7.04
640	0.25E-09	6.31	0.29E-10	7.18
1280	0.30E-11	6.38	0.16E-12	7.43

Table 3: Rate of Convergence At Third Order Critical Points for $P=2, \epsilon = \Delta x^m$.

N	WENO-Z		WENO-Z$_{cp}$	
	Error	Order	Error	Order
80	0.15E-05	7.14	0.15E-06	7.65
160	0.79E-08	7.61	0.49E-08	4.96
320	0.15E-09	5.68	0.15E-09	4.98
640	0.48E-11	4.99	0.48E-11	4.99
1280	0.15E-12	5.00	0.15E-12	5.00

5.1. 1D Numerical Results

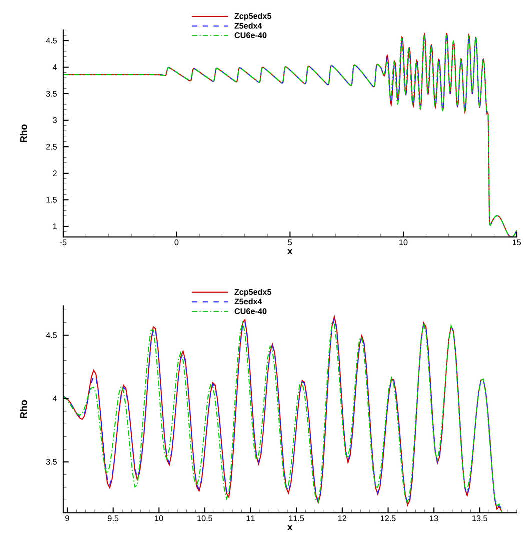

Figure 1: Shock Density Problem.

Fig. 1 shows the Shock Density Problem computed by the classic WENO-Z scheme, WENO-Z$_{cp}$ scheme and WENO-CU6 scheme. As we can observe from the results, our WENO-Z$_{cp}$ scheme is better than the other schemes.

5.2. 2D Numerical Results

WENO-Z5edx4 WENO-Z$_{cp}$5edx5

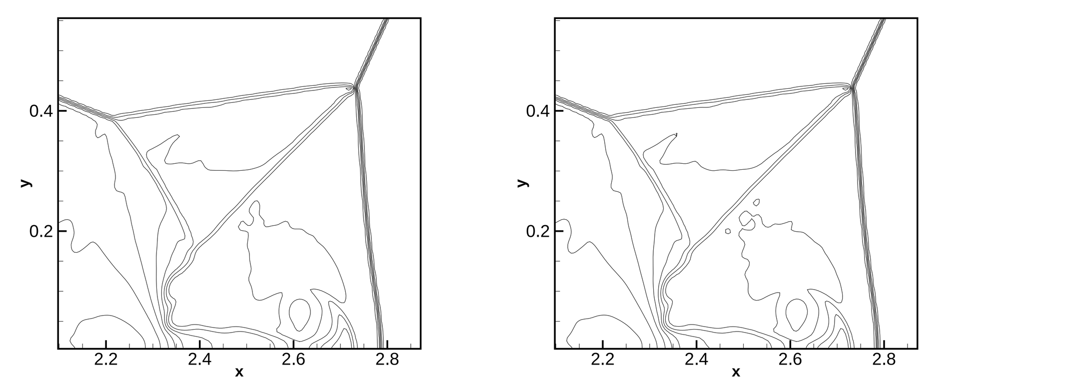

Figure 2: DMR Problem.

Fig. 2 shows the DMR Problem computed by the classic WENO-Z scheme and WENO-Z$_{cp}$ scheme. The WENO-Z$_{cp}$ scheme can capture more small structure than the classic WENO-Z scheme.

WENO-Z5edx4 WENO-Z$_{cp}$5edx5

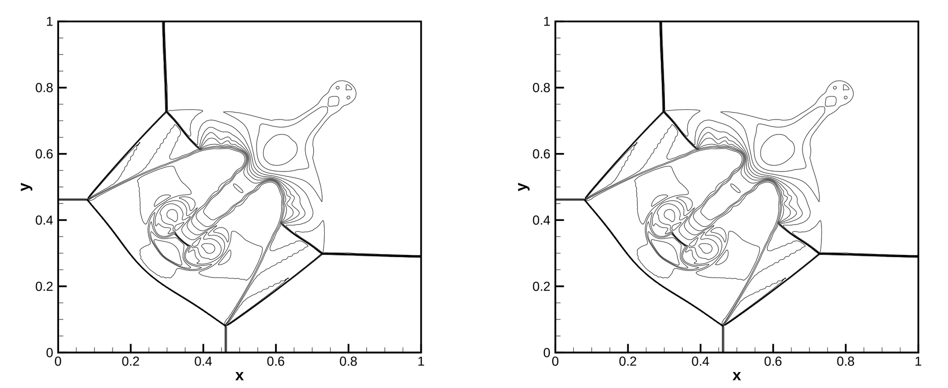

Figure 3: Riemann3 Problem.

Fig. 3 shows the Riemann3 Problem computed by the classic WENO-Z scheme and WENO-Z$_{cp}$ scheme. The New WENO-Z scheme can capture more small structure than the classic WENO-Z scheme.

References

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