Generalized Sensitivity Parameter Free Fifth Order WENO Finite Difference Scheme with Z-Type Weights

Research Center in Applied Mathematics and Scientific Computing

YingHua Wang, College of Oceanic and Atmospheric Sciences, Ocean University of China, Qingdao, China.

1. Abstract

A modified fifth order Z-type (nonlinear) weights in the weighted essentially non-oscillatory (WENO) polynomial reconstruction procedure for the WENO-Z finite difference scheme in solving hyperbolic conservation laws is proposed. The WENO scheme with the modified Z-type weights (WENO-D) scheme and its improved version (WENO-A) scheme are proposed. The analysis and numerical experiments show that, they achieve the optimal (fifth) order of accuracy regardless of the order of critical point with an *arbitrary small sensitivity parameter*, a-ka, satisfy the Cp-property. Furthermore, with an optimal variable sensitivity parameter, they have a quicker convergence and a significant error reduction over the WENO-Z scheme. The performance of the WENO schemes, in terms of resolution, essentially non-oscillatory shock capturing and efficiency, are compared by solving several one- and two-dimensional benchmark shocked flows. The results show that they perform overall as well as, if not slightly better than, the WENO-Z scheme.

2. The WENO-Z Scheme

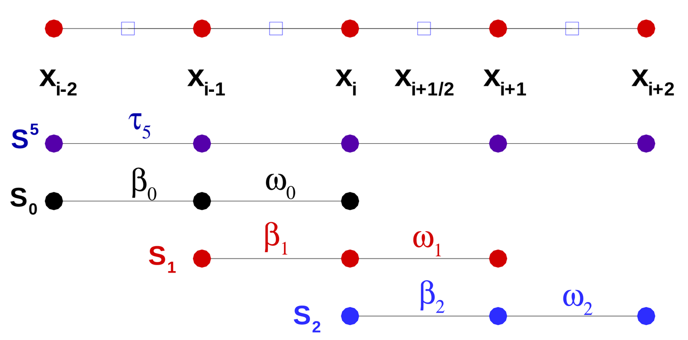

Figure 1: The computational uniformly spaced grid, with cell centers x_i and cell boundaries $x_{i+\frac{1}{2}}$, and the 5-points stencil S^5, composed of three 3-points substencils S_0, S_1, S_2, used in the fifth-order WENO reconstruction step.

In the $(2r-1)$ order WENO scheme with Z-type weights (WENO-Z), the nonlinear weights are

$$\alpha_k = d_k\left(1+\left(\frac{\tau_{2r-1}}{\beta_k+\varepsilon}\right)^p\right), \quad \omega_k^z = \frac{\alpha_k}{\sum_{j=0}^{r-1}\alpha_j}, \quad k=0,\ldots,r-1.$$

where the global smoothness indicator is

$$\tau_{2r-1} = \left|\sum_{k=0}^{r-1} c_k\beta_k\right|,$$

where c_k are given constants[a]. For example, τ_5 of the fifth order WENO-Z scheme is

$$\tau_5 = |\beta_0 - \beta_2|\,.$$

Its leading truncation error has been shown to be $O(\Delta x^5)$.

[a]Castro. Costa. and Don. J. Comput. Phys. 230, 1766–1792, 2011

3.1. The Cp-property

For any given power parameter p and sensitivity parameter ε, a WENO scheme is said to be satisfying the Cp-property if the WENO scheme achieves its optimal order of accuracy in approximating the first derivative of a smooth function regardless of critical points of any order.

3.2. The WENO-D/A Schemes

The generalized sensitivity parameter free fifth order WENO scheme with Z-type weights, which can satisfies the Cp-property. The nonlinear weights of the WENO-D scheme are

$$\alpha_k = d_k\left(1+\Phi\left(\frac{\tau_{2r-1}}{\beta_k+\varepsilon}\right)^p\right), \quad \omega_k = \frac{\alpha_k}{\sum_{j=0}^{r-1}\alpha_j}, \quad k=0,\ldots,r-1.$$

Based on the WENO-D scheme, a slight modification on the Z-type weights is applied to the nonlinear weights α_k, more specifically,

$$\alpha_k = d_k\left(\max\left(1,\Phi\left(\frac{\tau_{2r-1}}{\beta_k+\varepsilon}\right)^p\right)\right).$$

We refer this improved WENO-D scheme as WENO-A scheme.

4. Cp-property of the WENO-D/A Schemes

Table 1 gives the order of the nonlinear term Γ $\left(\Gamma = \phi\left(\frac{\tau_{2r-1}}{\beta_k+\varepsilon}\right)^p\right)$ in term of $\Delta(x)$ with $\varepsilon = 0$.

Table 1: The order of the nonlinear term with the Z-type weights for the fifth order WENO-D scheme, with $n_{cp} = 0, 1, 2, 3$, $p = 1, 2, 3$ and $\varepsilon = 0$.

				$\theta\left(\phi\left(\frac{\tau_{2r-1}}{\beta_k}\right)^p\right)$		
n_{cp}	$\theta(\phi)$	$\theta(\tau_5)$	$\theta(\beta_k)$	$p=1$	$p=2$	$p=3$
0	2	5	2	5	8	11
1	3	5	4	4	5	6
2	3	7	6	4	5	6
3	4	9	8	5	6	7

- The modified Z-type weights satisfy the sufficient condition $\Delta(\omega_k) = O(\Delta(x^3))$ automatically, hence, also the Cp-property.
- This implies that ε plays no role in determine the order of accuracy and can be chosen to be any small positive number (for example, $\varepsilon = 10^{-40}$) for avoiding the division of zero in the denominator in the nonlinear weights' definition.
- In this sense, the WENO-D/A schemes are truly sensitivity parameter free.

5. Convergence at Critical Points

Consider the following test function

$$f(x) = x^k e^{0.75x}, \quad x \in [-1,1],$$

in which its first $k-1$ derivatives $f^{(j)}(0) = 0, j = 0, 1, \ldots, k-1$. That is, this function has a critical point of order $n_{cp} = k-1$ at $x = 0$.

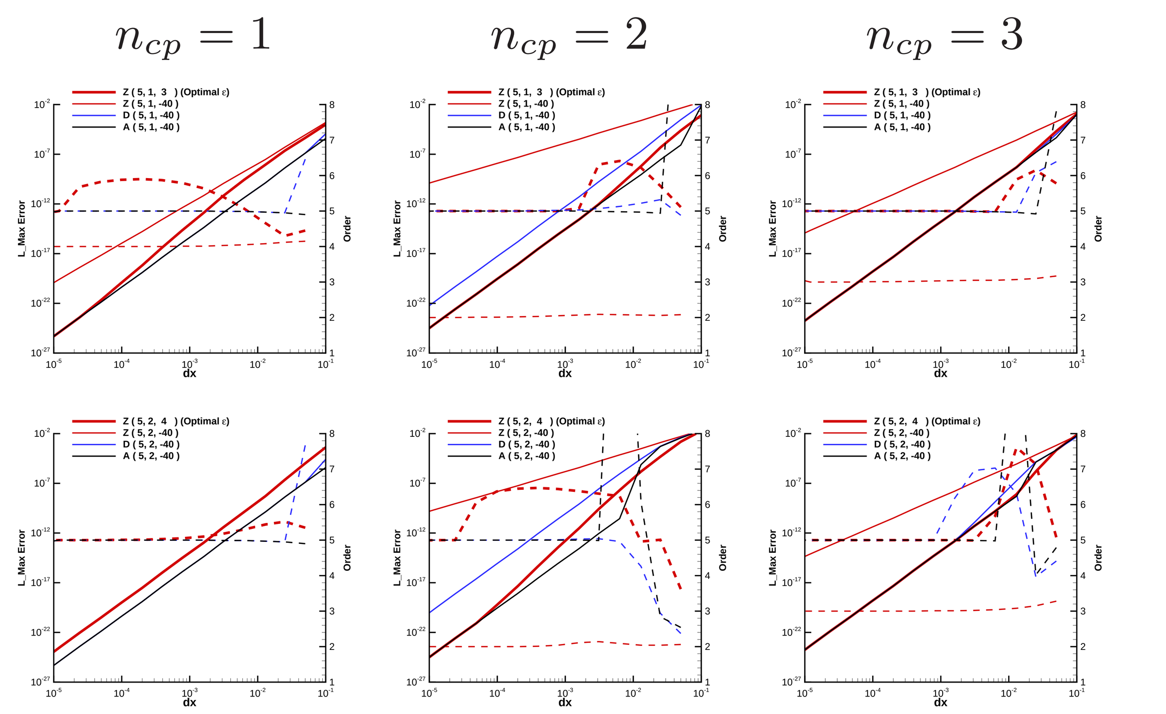

Figure 2: Constant $\varepsilon = 10^{-40}$ with (top row) $p = 1$ and (bottom row) $p = 2$: L_∞ error (solid lines) and order of accuracy (dotted lines).

6.1. 1D Numerical Results

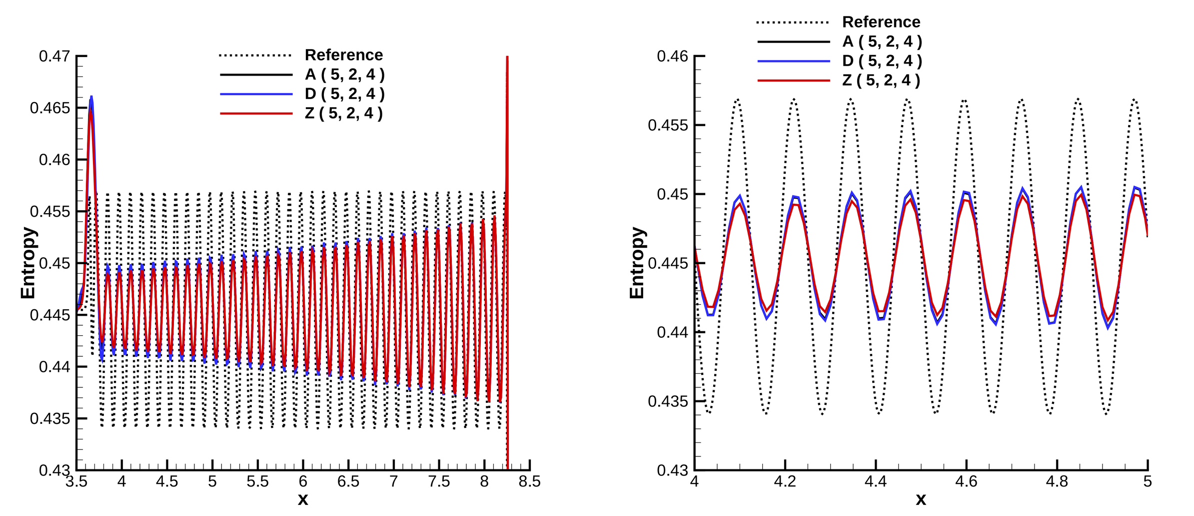

Figure 3: Entropy of the Mach 3 shock-entropy wave interaction computed by the WENO schemes at the final time $t = 5$. The number of mesh cells is $N = 1700$.

6.2. 2D Numerical Results

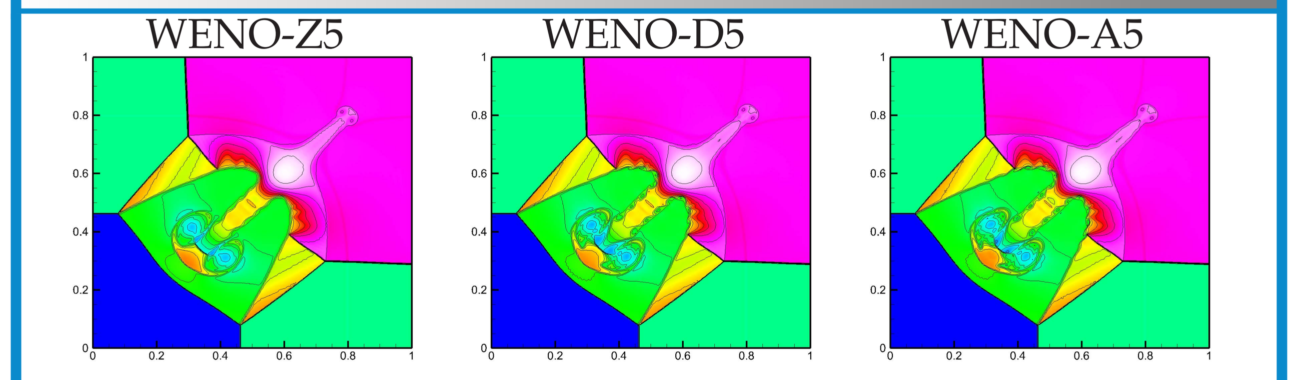

Figure 4: Density of the configuration 3 of the Riemann initial value problem computed by the WENO schemes with a variable $\varepsilon = \Delta(x^4) \approx 4 \times 10^{-11}$ at the final time $t = 0.8$. The number of mesh cells is $N \times M = 400 \times 400$.

A careful examination of these results reveals that, due to a lesser numerical dissipation, the WENO-A scheme presents a higher resolution of the structure of discrete vortices along the slip line than the WENO-Z and WENO-D schemes.

6.2. 2D Numerical Results

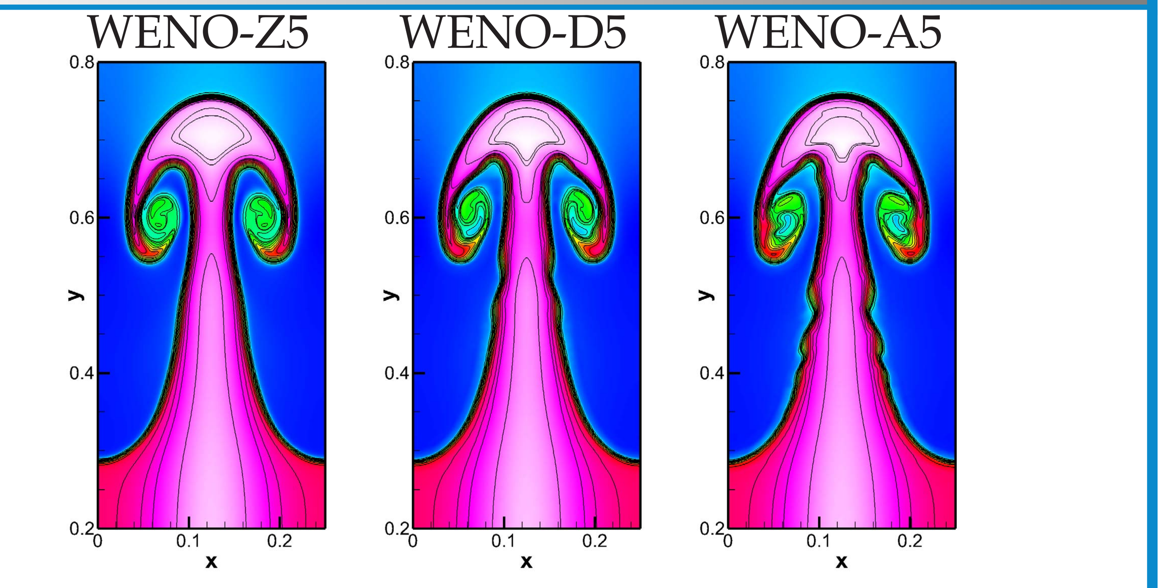

Figure 5: Solution of the RTI problem computed by the WENO schemes with $\varepsilon = \Delta(x^4) \approx 2 \times 10^{-11}$ at the final time $t = 1.95$. The number of mesh cells is $N \times M = 120 \times 480$.

Among the three schemes, the density computed by the WENO-A scheme resolves the small complex structures better than the other two schemes.

6.3. CPU Timings of the WENO Schemes

Table 2: The CPU time (in seconds) of the Riemann problem, the DMR problem and the RTI problem as computed by the WENO schemes where ($ratio$) is the ratio of CPU times between the proposed schemes and the WENO-Z scheme.

	$N \times M$	WENO-Z5	WENO-D5	WENO-A5
Riemann	400×400	2.7E+03	2.8E+03 (1.03)	2.7E+03 (1.00)
RTI	480×120	2.4E+03	2.5E+03 (1.04)	2.4E+03 (1.00)
DMR	800×200	3.1E+03	3.2E+03 (1.03)	3.2E+03 (1.03)

The timing shows that the WENO-D/A schemes are as efficient as the WENO-Z scheme.

References

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