Research on Some Kinds of CFDS

The Third Summer Workshop in Advanced Research in Applied Mathematics and Scientific Computing 2015, OUC, China

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Introduction

Compact finite difference scheme (CFDS), has advantages of high precision, high resolution, less demanding on the number of grid nodes.

In this poster, we mainly show and compare some kinds of compact difference schemes for the first derivative, and show the error and convergence order with an example.

We mainly show four CFDSs, including centered compact scheme, centered compact scheme with spectral-like resolution and two CFDSs for non-uniform grid. We give L^∞ error of the four schemes and convergence order of the two schemes for uniform grid by numerical experiments.

CFDS for uniform grid

We start our work from the simplest (and most familiar) finite difference scheme for first derivative, which is named centered compact scheme (CCS) and proposed by Lele [1]. Then, we'll show the basic idea in trying to design new central compact schemes.

CCS has the following form

$$\alpha u'_{i+1} + u'_i + \alpha u'_{i-1} = a\frac{u_{i+1}-u_{i-1}}{2h} + b\frac{u_{i+2}-u_{i-2}}{4h} \quad (1)$$

If the values at both the grid points and half grid points are used, we could derive a compact scheme with higher order accuracy and better resolution. Following this idea, we can design a class of central compact schemes with spectral-like resolution(CCSSR), as follows

$$\alpha u'_{i+1} + u'_i + \alpha u'_{i-1} = a\frac{u_{i+\frac{1}{2}}-u_{i-\frac{1}{2}}}{h} + b\frac{u_{i+1}-u_{i-1}}{2h} \quad (2)$$

The argument α, a, b can be derived from Taylor expansion according to the accuracy needed. The fourth-order CCS and CCSSR, respectively, are given by the following formula:

$$\frac{1}{4}u'_{i+1} + u'_i + \frac{1}{4}u'_{i-1} = \frac{3}{4h}(u_{i+1}-u_{i-1})$$

$$\frac{1}{22}u'_{i+1} + u'_i + \frac{1}{22}u'_{i-1} = \frac{12}{11}\frac{u_{i+\frac{1}{2}}-u_{i-\frac{1}{2}}}{h}$$

When $\alpha = \frac{1}{3}, a = \frac{14}{9}, b = \frac{17}{18}$ in (1), we get the sixth-order CCS

$$\frac{1}{3}u'_{i+1} + u'_i + \frac{1}{3}u'_{i-1} = \frac{14}{9}\frac{u_{i+1}-u_{i-1}}{2h} + \frac{1}{9}\frac{u_{i+2}-u_{i-2}}{4h}$$

When $\alpha = -\frac{1}{12}, a = \frac{16}{9}, b = \frac{17}{18}$ in (2), we get the sixth-order CCSSR

$$-\frac{u'_{i+1}}{12} + u'_i - \frac{u'_{i-1}}{12} = \frac{16}{9}\frac{u_{i+\frac{1}{2}}-u_{i-\frac{1}{2}}}{h} - \frac{17}{18}\frac{u_{i+1}-u_{i-1}}{2h}$$

CFDS for non-uniform grid

We derive the compact schemes using polynomial interpolation. We use Hermite-Birkhoff interpolation to obtain an explicit form of compact difference schemes.
The fourth-order accurate first derivative compact scheme has the following form

$$\left(\frac{x_i-x_{i+1}}{x_{i-1}-x_{i+1}}\right)^2 u'_{i-1} + u'_i + \left(\frac{x_i-x_{i-1}}{x_{i+1}-x_{i-1}}\right)^2 u'_{i+1}$$
$$= 2\left(\frac{1}{x_i-x_{i-1}} + \frac{1}{x_i-x_{i+1}}\right)u_i +$$
$$2\left(\frac{x_i-x_{i+1}}{x_{i-1}-x_{i+1}}\right)^2\left(\frac{1}{x_{i-1}-x_i} + \frac{1}{x_{i-1}-x_{i+1}}\right)u_{i-1} +$$
$$2\left(\frac{x_i-x_{i-1}}{x_{i+1}-x_{i-1}}\right)^2\left(\frac{1}{x_{i+1}-x_i} + \frac{1}{x_{i+1}-x_{i-1}}\right)u_{i+1}$$

For example, fourth-order accurate first derivative compact scheme for a non-uniform grid with the distribution of nodes given by $x_i = x_1 + h(i-1)^2, i = 1, 2, ..., N$ is presented below (SCHEME3):

$$\frac{(2i-1)^2}{16(i-1)^2}u_{i-1} + u_i + \frac{(2i-3)^2}{16(i-1)^2}u_{i+1}$$
$$= \frac{4}{(2i-3)(2i-1)h}u_i - \frac{(2i-1)^2}{8h(i-1)^2}\left(\frac{1}{2i-3} +\right.$$
$$\left.\frac{1}{4(i-1)}\right)u_{i-1} + \frac{(2i-3)^2}{8h(i-1)^2}\left(\frac{1}{2i-1} + \frac{1}{4(i-1)}\right)u_{i+1}$$

Another example, the whole domain can be segmented into some small domains, in each of which we have uniform grids. Following this idea, the distribution of nodes are given (as Fig.1 shows) by

$$\begin{cases} x_1 = x_0 + h_1(i-1), i = 1, ..., N1 \\ x_i = x_{i-1} + h_2, i = N_1+1, ..., N_1+N_2 \\ x_i = x_{i-1} + h_1, i = N_1+N_2+1, ..., 2N_1+N_2 \end{cases}$$

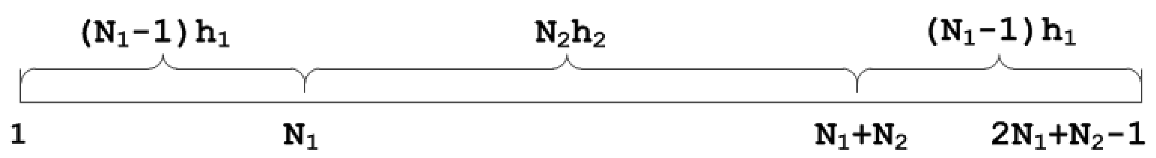

Figure 1: The distribution of nodes for SCHEME4.

Then, at index $i = N_1$, the fourth-order accurate CFDS for non-uniform grid (SCHEME4) has form below

$$\frac{h_2^2}{(h_1+h_2)^2}u'_{i-1} + u'_i + \frac{h_1^2}{(h_1+h_2)^2}u'_{i+1}$$
$$= 2\left(\frac{1}{h_1} - \frac{1}{h_2}\right)u_i - \frac{2h_2^2}{(h_1+h_2)^2}\left(\frac{1}{h_1} - \frac{1}{h_1+h_2}\right)u_{i-1} +$$
$$\frac{2h_1^2}{(h_1+h_2)^2}\left(\frac{1}{h_2} - \frac{1}{h_1+h_2}\right)u_{i+1}$$

And at index $i = N_1 + N_2$, the form is similar to index $i = N_1$. At other index, SCHEME4 has forms similar to CCS.

Numerical experiments for uniform grid

Take $u = \sin(x), x \in [0, 2\pi]$, for example. The first derivative is $u' = \cos(x), x \in [0, 2\pi]$. On the boundary, we use accurate first derivative.

Table 1 and 2 respectively shows L^∞ error and convergence order of using CCS and CCSSR with non-periodic problem (NPP). Here N means number of points.

Table 1: Results of CCS solving NPP

N	L^∞ error	Order	L^∞ error	Order
20	2.17E-05	-	1.12E-07	-
40	1.36E-06	4.0008	1.76E-09	5.9987
80	8.48E-08	4.0002	2.75E-11	5.9997
160	5.30E-09	4.0001	4.29E-13	6.0003

Table 2: Results of CCSSR solving PP

N	L^∞ error	Order	L^∞ error	Order
20	2.88E-05	-	3.48E-06	-
40	1.80E-06	4.0009	5.65E-08	5.9466
80	1.12E-07	4.0003	8.91E-10	5.9861
160	7.02E-09	4.0001	1.40E-11	5.9972

Table 3 and 4 respectively shows L^∞ error and convergence order of using CCS and CCSSR to solve periodic problem (PP).

Table 3: Results of CCS solving PP

N	L^∞ error	Order	L^∞ error	Order
20	5.48E-05	-	4.63E-07	-
40	3.39E-06	4.0127	7.71E-09	6.0125
80	2.12E-07	4.0031	1.12E-10	6.0031
160	1.32E-08	4.0002	1.76E-12	5.9939

Table 4: Results of CCSSR solving PP

N	L^∞ error	Order	L^∞ error	Order
20	2.88E-05	-	5.65E-08	-
40	1.80E-06	4.0003	8.91E-10	5.9861
80	1.12E-07	4.0001	1.40E-11	5.9957
160	7.02E-09	4.0000	2.41E-13	5.8587

The four tables shows that errors of CCS are larger then these of CCSSR.

Numerical experiments for non-uniform grid

Using the same example and boundary condition, we give L^∞ error for using SCHEME3 and SCHEME4 to solve non-periodic problem (NPP).

In SCHEME3, we can easily arrive at h according to the value of N. And in SCHEME4, we give values of $rate_1$ and $rate_2$: $rate_1 = \frac{N_1}{2N_1+N_2} = 0.25$ and $rate_2 = \frac{(N_1-1)h_1}{2\pi} = 0.1$. So we can easily arrive at N_1 and h_1 according to the value of $2N_1 + N_2$. Table 5 shows L^∞ error of using the two schemes for non-uniform grid.

Table 5: L^∞ error for non-uniform grid

		SCHEME3	SCHEME4
N or $2N_1 + N_2$	h	L^∞ error	L^∞ error
20	0.017	3.13E-04	5.41E-04
40	0.004	1.57E-04	2.72E-05
80	0.001	1.61E-05	1.53E-06
160	2.48E-4	2.28E-06	9.11E-08

Future Work

1.Analyze the error and resolution of CFDS for non-uniform grid.
2.Give a scheme of higher order or resolution according to the analysis.
3.Reserch on CFDSs for higher derivative.

References

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Acknowledgement

I wish to express my sincere gratitude to all those who have offered me invaluable help during summer workshop. This work is supported by funding that is described in the following section. At first, I gratefully acknowledge the financial and academic support by Prof. Fang Qizhi, Prof. Gao Zhen, Prof. Xie, Shusen and Prof. Don Wai Sun. Second, I would like to express my heartfelt gratitude to Mr. Zhu Qiangqiang and Mr. Li Peng who helped me a lot. Also, I'd like to give thanks to my partners and my family.

Funding

- School of Mathematical Sciences, OUC.
- Startup grant by Ocean University of China (Don).
- Natural Science Foundation of Shandong Province (ZR2012AQ003).
- National Natural Science Foundation of China (11201441).