RKDG Method to multi-dimensional systems of conservation laws

The Third Summer Workshop in Advanced Research in Applied Mathematics and Scientific Computing 2015, OUC, China

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Introduction

Discontinuous Galerkin (DG) methods are a class of finite element methods using discontinuous basis functions, such as piecewise polynomials. We consider the method to solve the Euler equations and study how to use the local projection limiting in the characteristic fields. Numerical experiments for 1-D Riemann problem and 2-D double Mach reflection problem on rectangular elements are presented.

DG Scheme

The RKDG method is a method devised to numerically solve the initial boundary value problem associated with the conservation law

$$\partial_t Q + \nabla \mathbf{f}(Q) = 0 \; in \; \Omega \times (0,T), \tag{1}$$

where $\Omega \subset R^d$ and $Q = (Q_1, \ldots, Q^m)^t$, Which is assumed to hyperbolic ; that is ,$\mathbf{f}(Q)$ is assumed to be such that any real combination of the Jacobians $\sum_{i=1}^{d} \xi_i(\partial f_i/\partial Q)$has m real eigenvalues and a complete set of eigenvectors.

We extend (and improve) the RKDG method to the case of multidimensional systems from 1D scalar equation, first we discretize (1) in space using the discontinuous Galekin method and we use TVD Runge-Kutta method to discretize time. To determine the approximate solution $u_h(t)$, we need the weak formulation of (1):

$$\frac{d}{dt}\int_K Q v dx = -\int_e \mathbf{f}(Q)\cdot n_{e,K} v d\Gamma + \int_K \mathbf{f}(Q)\cdot \nabla v dx,$$

for any smooth function $v(x,y)$. Here $n_{e,K}$denotes the outward unit normal of the edge e.

We replace the integrals by quadrature rules as

$$\int_e \mathbf{f}(Q)\cdot n_{e,K} v(x) d\Gamma \simeq \sum_{l=1}^{L} \omega_l \mathbf{f} Q \cdot n_{e,K} v(x_{el})|e|, \tag{2}$$

$$\int_K \mathbf{f}(Q)\cdot \nabla v dx \simeq \sum_{j=1}^{M} \Omega_j \mathbf{f} Q_{K_j} \cdot \nabla v_{K_j} |K|. \tag{3}$$

Then, the flux $\mathbf{f}(Q)\cdot n_{e,K}$ is replaced by the numerical flux $h_{e,K}(x,t)$

$$h_{e,K}(x,\cdot) = h_{e,K}(u_h(x^{int(K)},\cdot), u_h(x^{ext(K)},\cdot)), \tag{4}$$

DG Scheme

where $h_{e,K}$ is any two-point Lipschitz flux which is monotone in the scalar case and is an exact or approximate Riemann solver in the system case. The value of the numerical flux depends on the two values of the approximate solution at $(\mathbf{x},t)$ which are

$$Q_h(\mathbf{x}^{int(K)}) = \lim_{\mathbf{y}\to\mathbf{x}, \mathbf{y}\in K} Q_h(\mathbf{y},t), \tag{5}$$

and

$$Q_h(\mathbf{x}^{ext(K)}) = \begin{cases} \gamma_h(\mathbf{x},t), if \; \mathbf{x} \in \partial\Omega, \\ \lim_{\mathbf{y}\to\mathbf{x}, \mathbf{y}\notin K} Q_h(\mathbf{y},t), \text{otherwise}. \end{cases}$$

Minmod Limiter

For 2D case, we just need to limit the mean value on each element. Here we denote the P2 approximation solution as follows:

$$Q_h(x,y,t) = \bar{Q} + \sum_{i=1}^{5} Q_i \phi_i, \tag{6}$$

For systems, we perform the limiting in the local characteristic variables. To limit the vector Q_1 in the element .We proceed as follows: Find the matrix R and its inverse R^{-1}, which diagonalize the Jacobian evaluated at the mean in the element in the $x-$direction,

$$\frac{\partial \mathbf{f}}{\partial Q} = R^{-1}\Lambda R, \tag{7}$$

and we should transform all quantities needed for limiting. This is achieved by left-multiplying these three vectors by R^{-1}. After the Minmod function ,we transformed back to the original space by left multiplying R on the left. Next, we introduce the limiter for P2, P3, or higher order.First we compute:

$$Q_k^{mod} = \begin{cases} Q_k, \;\; if \mid Q_k \mid \le M_{lim} \\ minmod(Q_k, \; \gamma\triangle_{i+1}Q, \; \gamma\triangle_i Q, \; otherwise \end{cases} \tag{8}$$

The parameter M_{lim} and γ control the amount of limiting,it usually take :

$$M_{lim} = M\Delta x^2, \;\; M = 50.0, \;\; \gamma = 0.5 \; or \; 1.0 \tag{9}$$

If $Q_1^{mod} \neq Q_1, Q_2^{mod} \neq Q_2$, the solution Q_h is replaced by

$$\mathcal{L}Q_h = \bar{Q} + Q_1^{mod}\phi_1 + Q_2^{mod}\phi_2, \tag{10}$$

1-D Riemann Problem

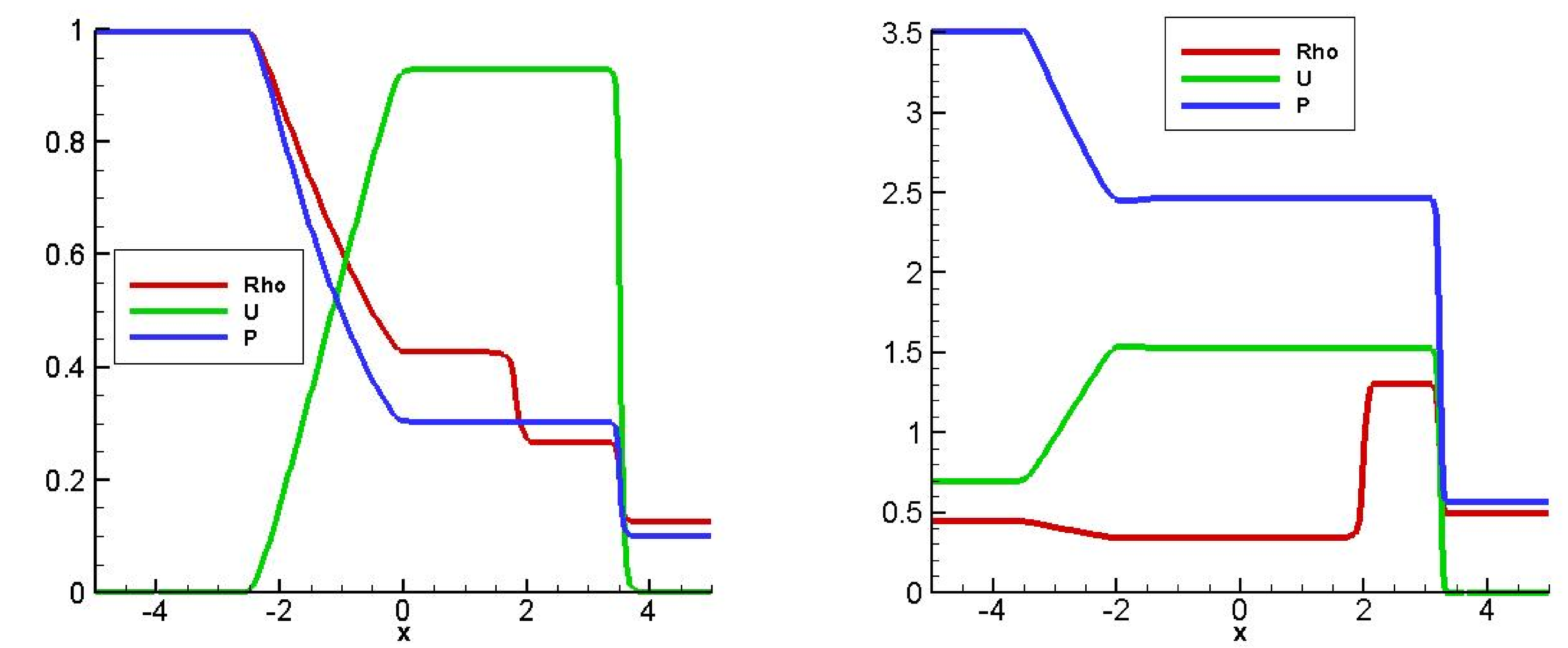

Figure 1: 1-D Riemann Problems as computed by P^2-RKDG scheme with characteristic-wise limiters and $M_2 = 0$.

In Fig. 1, we use equally spaced cells and the results show that the RKDG scheme performs well for these problems with a much reduced non-physical oscillations.

2-D Euler Equation

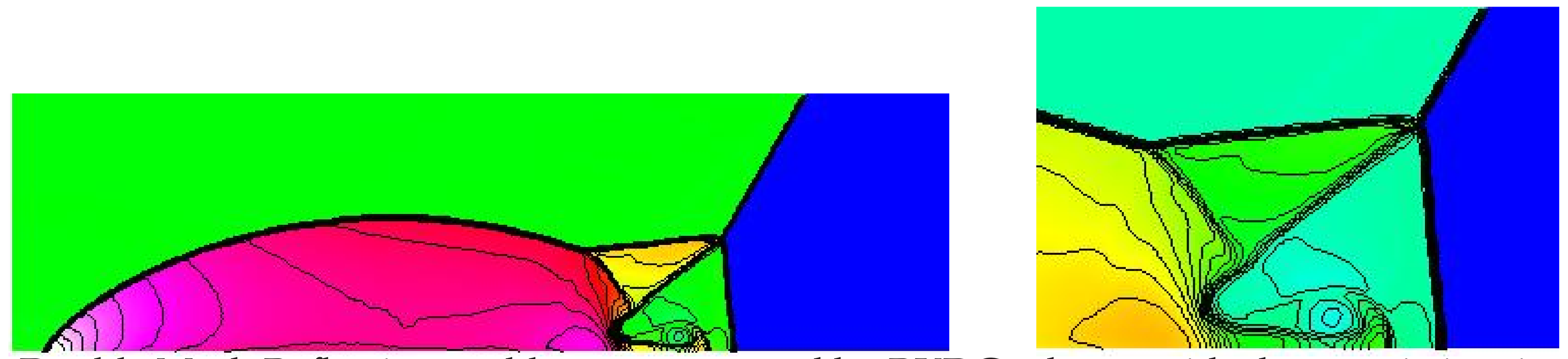

Figure 2: Double Mach Reflection problem as computed by RKDG scheme with characteristic-wise limiters.

In Fig. 2, we use uniform rectangular elements with $\Delta x = \Delta y = \frac{1}{120}$. The left one is P1 case, the Right one is P2 case.

References

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Future Work

- Extension of RKDG scheme to 2D Euler equation by using non-uniform triangular elements.
- Extension of RKDG scheme to 2D Euler equation by using triangular elements.

Acknowledgement

I owe my sincere gratitude to Prof. Fang Qizhi, Mr. Li Peng, Prof. Gao Zhen, Prof. Xie, Shusen and Prof. Don Wai Sun for the financial and academic support. At the same time, special thanks should go to my friends who have helped me directly and indirectly in my studies.

Funding

School of Mathematical Sciences, OUC. Startup grant by Ocean University of China (201412063). Natural Science Foundation of Shandong Province (ZR2012AQ003). National Natural Science Foundation of China (11201441).