Stochastic Collocation Method and Reduced Basis Method

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Introduction

In recent years, there has been a growing need for including uncertainty in mathematical models and quantify its effect on given outputs of interest used in decision making. In general, a probabilistic setting can be used to include these uncertainties in mathematical models. In such framework, the random input data are modeled as random variables, or random fields with a given correlation structure. Thus, the goal of the mathematical and computational analysis becomes the prediction of statistical moments of the solution or statistics of some quantities of physical interest of the solution, given the probability distribution of the input random data. Stochastic collocation (SC) method is a non-intrusive method for uncertainty quantification (UQ), it can be easily implemented and leads to the solution of uncoupled deterministic problems, even in presence of input data which depend nonlinearly on the driving random variables.

In such models a number of random input-parameters are used to characterize a particular problem which require repeated evaluation, so alternative method to simply solving the full problems many times is needed, that is, reduced basis method (RBM). The goal of RBM is to approximate a set of given data with a low number of basis functions, where the proper orthogonal decomposition (POD) is a well-known method to obtain the basis.

Random differential equation

Consider a d-dimensional bounded domain $D \subset \mathbb{R}^d$ $(d = 1,2,3)$ with boundary ∂D, and the random differential equations: find a stochastic function, $u \equiv u(\omega, x) : \Omega \times \bar{D} \to \mathbb{R}$, such that for $\forall \omega \in \Omega$, the following equation holds:

$$\mathfrak{L}(\omega, x; u) = f(\omega, x), \quad x \in D, \tag{1}$$

subject to the boundary condition

$$\mathfrak{B}(\omega, x; u) = g(\omega, x), \quad x \in \partial D, \tag{2}$$

where $x = (x_1, \cdots, x_d)$ are the coordinates in $\mathbb{R}^d$. The strong form is

$$\mathfrak{L}(y, x; u) = f(y, x), \quad (y, x) \in \Gamma \times D, \tag{3}$$

subject to the boundary condition

$$\mathfrak{B}(y, x; u) = g(y, x), \quad (y, x) \in \Gamma \times \partial D, \tag{4}$$

where $\Gamma \equiv \prod_{i=1}^N \Gamma^i \subset \mathbb{R}^N, \Gamma^i = Y^i(\Omega)$.

SC methods

Let $\Theta_N = \{y_i\}_{i=1}^M$ be a set of (prescribed) nodes such that the Lagrange interpolation in the N-dimensional random space Γ is poised. By denoting

$$\hat{u}(y) \equiv \mathcal{I}u(y) = \sum_{k=1}^{M} u(y_k) L_k(y), \tag{5}$$

where $L_k(y)$ is the lagrange polynomials. The stochastic collocation procedure is to solve M deterministic problems

$$R(\hat{u}(y)) \mid_{y_k} = 0, \quad \forall k = 1, \cdots, M, \tag{6}$$

where $R(u) = \mathfrak{L}(u) - f$ is the residual of (3). By using the property of Lagrange interpolation, we obtain y_k satisfies (3)-(4), for $k = 1, \cdots, M$.

The mean of the random solution is

$$\mathbb{E}(\hat{u})(x) = \sum_{k=1}^{M} u(y_k, x)\omega_k, \tag{7}$$

where ω_k is the gaussian quadrature weights.

Reduced Basis Methods

The parametric problem given as: find $u(\mu) \in \mathbb{V}$, such that

$$a(u(\mu), v; \mu) = f(v; \mu), \quad \forall v \in \mathbb{V}.$$

The solution manifold is $\mathcal{M} = \{u(\mu) | \mu \in \mathbb{P}\} \in V$.

The goal of RBM is to approximate any member of solution manifold with a low number of, say N, basis functions $\{\xi_n\}_{n=1}^N \subset \mathbb{V}_\delta$. Given the $N-$ dimensional $(N \ll N_\delta)$ reduced basis space $\mathbb{V}_{rb} = span\{\xi_1, \cdots, \xi_N\}$, the reduced basis approximation is sought as: for any given $\mu \in \mathbb{P}$, find $u_{rb}(\mu) \in \mathbb{V}_{rb}$, s.t.

$$a(u_{rb}(\mu), v_{rb}; \mu) = f(v_{rb}; \mu), \quad \forall v_{rb} \in \mathbb{V}_{rb}, \tag{8}$$

and evaluate

$$s_{rb}(\mu) = f(u_{rb}(\mu); \mu), \quad \forall v_{rb} \in \mathbb{V}_{rb}. \tag{9}$$

Proper Orthogonal Decomposition (POD) is one of the methods to generate the reduced basis.

POD

Assuming that we have a set of K-dimensional vectors (say, M of them), collected in matrix U, the main idea of POD is to find N orthogonal basis $\{\xi_n\}_{n=1}^N$ to approximate u_i by

$$u_i \approx \sum_{j=1}^{N} a_{ij}\xi_j, \quad i = 1, \cdots, M,$$

where the orthogonal basis collected in matrix Φ.

POD

The orthogonal POD basis has columns defined by

$$\xi_i = U \cdot v_i \cdot \lambda_i^{-\frac{1}{2}}, \tag{10}$$

and sorted in descending order of eigenvalues λ_i in a matrix

$$\Phi = [\xi_i], \quad i = 1, \cdots, M, \tag{11}$$

where the v_i and λ_i is the eigenvectors and eigenvalues of the correlation matrix

$$D = U^T U,$$

and the coefficient matrix $A = \Phi^T U$.

Then, the approximation of U can be constructed as

$$U \approx \Phi_i A_i, \quad i = 1, \cdots, N, \tag{12}$$

where Φ_i denotes the first i columns of Φ, and the A_i denotes the first i rows of A.

stochastic ODE

First, we consider a simple ODE with random variables

$$\begin{cases} \frac{d}{dt}u(t, \omega) = -\alpha(\omega)u(t, \omega), & t \in (0, 1) \\ u(0, \omega) = \beta(\omega), & t = 0. \end{cases} \tag{13}$$

Here, we consider $\beta(\omega) \equiv 1$, and the corresponding quadrature nodes and weights are Gauss-Legendre quadrature and Hermite quadrature, respectively.

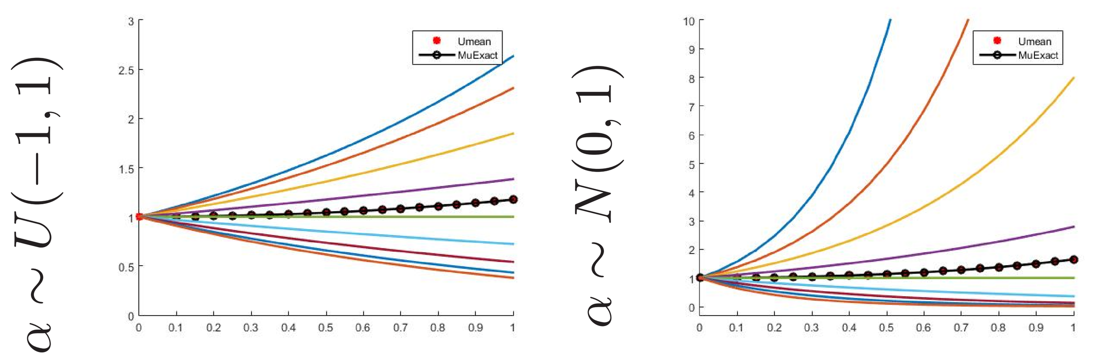

Figure 1: The deterministic solutions with $M = 9$ and the mean.

From Fig.1, it is seen that for time $t = 1$ there is a lot of uncertainty in the result. This uncertainty would be increased in time since the variance increases exponentially in time.

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Burgers equation

The one-dimensional viscous Burgers' equation with parameterized diffusion coeffcient is taken into consideration, given as

$$\begin{cases} u_t + uu_x - \nu u_{xx} = 0, \\ u(-1, t; \mu) = u(1, t; \mu) = 0, \\ u(x, 0; \mu) = -\sin(\pi x) \end{cases} \tag{14}$$

where $(x, t, \nu) \in (-1, 1) \times (0, 1] \times [0.0064, 0.0477]$.

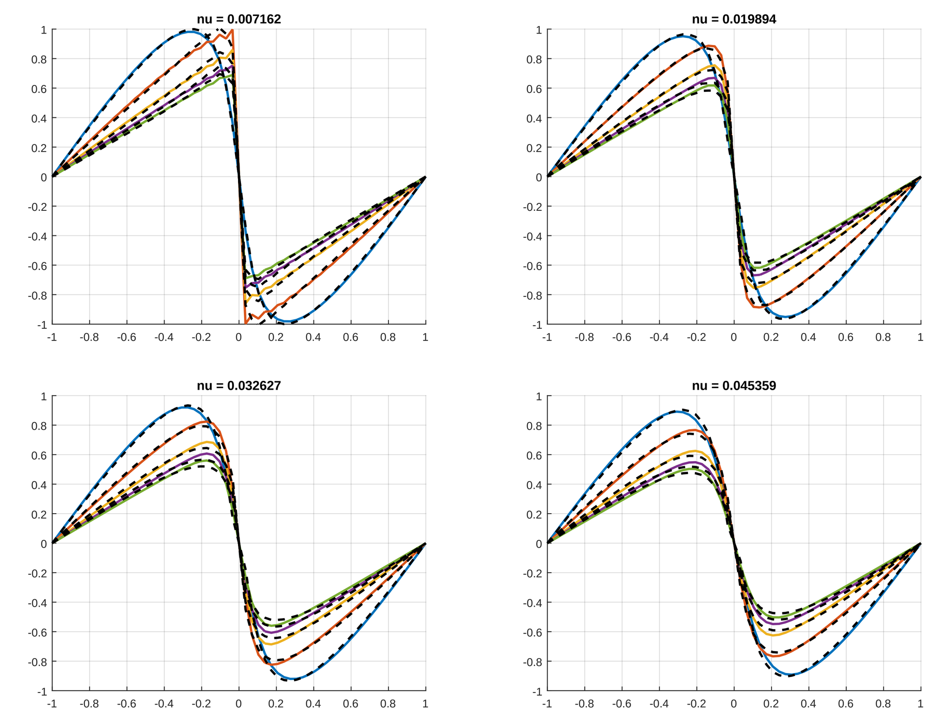

Figure 2: Comparison between the full-order solutions and the corresponding reduced-order solutions: solutions at different $t \in \{0.25, 0.50, 0.75, 0.90, 1.00\}$ with different values of ν: reduced-order solutions-dashed, full-order solution -solid.

References

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Future Work

- Extension to 2D problems.
- Realization of other numerical methods for UQ.
- Studying of Certified Reduced Basis method.