Fifth Order AWENO Finite Difference Scheme with Adaptive Numerical Diffusion

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1. Abstract

In solving nonlinear hyperbolic conservation laws, adaptive numerical diffusion is achieved through a modified local Lax-Friedrichs (LLF) flux based on the fifth-order characteristic-wise alternative WENO-Z finite-difference scheme (AWENO). The numerical diffusion coefficients of the LLF flux are adapted by a modification function. By the modification function, less numerical diffusion is used in the smooth regions of the solution to improve the resolution of the small-scale structures, and more numerical diffusion is adaptively used in the discontinuous regions to avoid numerical oscillation. In order to enhance the robustness of the numerical scheme, the adjustment range of the numerical diffusion coefficients are limited to a given upper and lower bound. Some one- and two-dimensional benchmark numerical examples with discontinuities show that the scheme can well reduce the numerical diffusion in the smooth regions, improve the resolution of the solution, and has strong ability to capture fine scale structures, while ensuring the robustness of the numerical scheme even in a long term simulation.

2.1. The WENO interpolation procedure

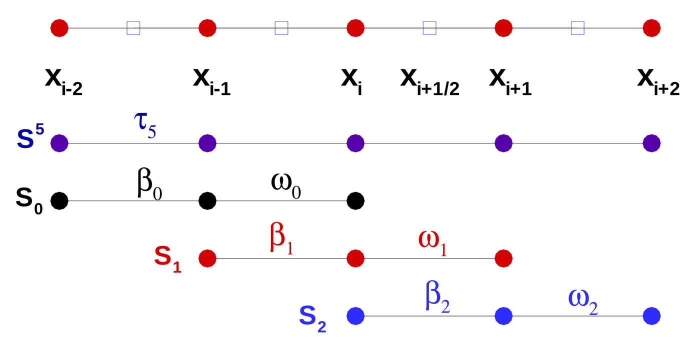

Figure 1: The computational uniformly spaced grid, with cell centers x_i and cell boundaries $x_{i+\frac{1}{2}}$, and the 5-points stencil S^5, composed of three 3-points substencils S_0, S_1, S_2, used in the fifth-order WENO reconstruction step.

In the fifth order WENO scheme with Z-type weights (WENO-Z), the nonlinear weights are

$$\alpha_k = d_k\left(1+\left(\frac{\tau_5}{\beta_k+\varepsilon}\right)^p\right), \quad \omega_k^z = \frac{\alpha_k}{\sum_{j=0}^{2}\alpha_j}, \quad k=0,...,2.$$

where $d_0 = \frac{1}{16}, d_1 = \frac{5}{8}, d_2 = \frac{5}{16}$ are the ideal weights which guarantees the fifth order (optimal) accuracy of the overall scheme. And the global optimal order smoothness indicator τ_5 is given as $\tau_5 = |\beta_0 - \beta_2|$,along with the local lower order smoothness indicators

$$\beta_0 = \frac{13}{12}(Q_{i-2}-2Q_{i-1}+Q_i)^2 + \frac{1}{4}(Q_{i-2}-4Q_{i-1}+3Q_i)^2,$$
$$\beta_1 = \frac{13}{12}(Q_{i-1}-2Q_i+Q_{i+1})^2 + \frac{1}{4}(Q_{i-1}-Q_{i+1})^2,$$
$$\beta_2 = \frac{13}{12}(Q_i-2Q_{i+1}+Q_{i+2})^2 + \frac{1}{4}(3Q_i-4Q_{i+1}+Q_{i+2})^2.$$

2.2. The Fifth-Order AWENO scheme

Consider the hyperbolic conservation laws with the formulation

$$\mathbf{Q}_t + \nabla\cdot\mathbf{F}(\mathbf{Q}) = 0,$$

Assume that $\mathbf{F}(\mathbf{Q})$ is a smooth function of $\mathbf{Q}(x)$ and need to find a consistent numerical flux $\hat{\mathbf{F}}(x)$ such that

$$\frac{1}{\Delta x}\left(\hat{\mathbf{F}}_{i+\frac{1}{2}} - \hat{\mathbf{F}}_{i-\frac{1}{2}}\right) = \left.\frac{d\mathbf{F}(\mathbf{Q}(x))}{dx}\right|_{x_i} + O(\Delta x^{2r-1}).$$

The fifth-order numerical flux $\hat{\mathbf{F}}_{i+\frac{1}{2}}$ is obtained by the sixth-order accurate Taylor expansion of $\mathbf{F}$ at $x = x_{i+\frac{1}{2}}$,

$$\hat{\mathbf{F}}_{i+\frac{1}{2}} = \mathbf{F}_{i+\frac{1}{2}} - \frac{1}{24}\Delta x^2\mathbf{F}_{xx}|_{i+\frac{1}{2}} + \frac{7}{5760}\Delta x^4\mathbf{F}_{xxxx}|_{i+\frac{1}{2}} + O(\Delta x^6).$$

The first term of the numerical flux is approximated by $\mathbf{F}_{i+\frac{1}{2}} = h(\mathbf{Q}^-_{i+\frac{1}{2}}, \mathbf{Q}^+_{i+\frac{1}{2}})$ with the values $\mathbf{Q}^{\pm}_{i+\frac{1}{2}}$ obtained by the fifth order WENO interpolation procedure which is applied to the *conservative variables* rather than the *flux functions*. The two-argument function $h(\mathbf{Q}^-, \mathbf{Q}^+)$ is a monotone flux. The LLF flux and its modification are used.
The second and fourth derivatives terms are

$$\Delta x^2 F_{xx}|_{i+\frac{1}{2}} = \frac{1}{48}(-5F_{i-2}+39F_{i-1}-34F_i - 34F_{i+1}+39F_{i+2}-5F_{i+3}) + O(\Delta x^6),$$
$$\Delta x^4 F_{xxxx}|_{i+\frac{1}{2}} = \frac{1}{2}(F_{i-2}-3F_{i-1}+2F_i + 2F_{i+1}-3F_{i+2}+F_{i+3}) + O(\Delta x^6).$$

3. The local Lax-Friedrichs (LLF) flux

The LLF flux is defined as

$$h^{\mathrm{LF}}(\mathbf{Q}^-_{i+\frac{1}{2}}, \mathbf{Q}^+_{i+\frac{1}{2}}) = \frac{1}{2}\left[\mathbf{F}(\mathbf{Q}^-_{i+\frac{1}{2}}) + \mathbf{F}(\mathbf{Q}^+_{i+\frac{1}{2}}) - \alpha_{i+\frac{1}{2}}(\mathbf{Q}^+_{i+\frac{1}{2}} - \mathbf{Q}^-_{i+\frac{1}{2}})\right]$$

where the numerical diffusion coefficients $\alpha_{i+\frac{1}{2}} = \max\left\{\left|u^+_{i+\frac{1}{2}}\right| + c^+_{i+\frac{1}{2}}, \left|u^-_{i+\frac{1}{2}}\right| + c^-_{i+\frac{1}{2}}\right\}$ is taken as the spectral radius of the Jacobian $A(\boldsymbol{Q}) = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{Q}}$ and its physical meaning is the maximum local propagation speed.

4.1. The LLF-M flux

The modified LLF flux is abbreviated as the LLF-M flux with the form

$$h^{\mathrm{LF-M}}(\mathbf{Q}^-_{i+\frac{1}{2}}, \mathbf{Q}^+_{i+\frac{1}{2}}) = \frac{1}{2}\left[\mathbf{F}(\mathbf{Q}^-_{i+\frac{1}{2}}) + \mathbf{F}(\mathbf{Q}^+_{i+\frac{1}{2}}) - \bar{\alpha}_{i+\frac{1}{2}}(\mathbf{Q}^+_{i+\frac{1}{2}} - \mathbf{Q}^-_{i+\frac{1}{2}})\right],$$

with $\bar{\alpha}_{i+\frac{1}{2}} = \kappa_{i+\frac{1}{2}}\alpha_{i+\frac{1}{2}}$.

4.2. The modification function

An important part of the modification function κ is the smoothness indicators derived from the WENO-η scheme, in which the lower order local smoothness indicators η_k (similar to β_k) are

$$\eta_0 = \frac{1}{4}(F_{i-2}-4F_{i-1}+3F_i)^2 + (F_{i-2}-2F_{i-1}+F_i)^2,$$
$$\eta_1 = \frac{1}{4}(F_{i-1}-F_{i+1})^2 + (F_{i-1}-2F_i+F_{i+1})^2,$$
$$\eta_2 = \frac{1}{4}(3F_i-4F_{i+1}+F_{i+2})^2 + (F_i-2F_{i+1}+F_{i+2})^2.$$

And the global higher order smoothness indicator is

$$\tau_5^\eta = |\eta_0 - \eta_2| + O(\Delta x^6).$$

$\kappa_{i+\frac{1}{2}} \in [\kappa_{\min}, 1]$ is the value of the modifier

$$\kappa = \begin{cases} \max(1-e^{(-\delta\tau_m^\gamma)}, \kappa_{\min}), & \text{if} \quad \tau_s \leqslant \eta_{\max} \\ 1, & \text{else.} \end{cases}$$

Where

$$\tau_m|_{x_{i+\frac{1}{2}}} = \tau_s|_{x_{i+\frac{1}{2}}} \Big/ \left(\sum_{i=1}^{N}\tau_s|_{x_{i+\frac{1}{2}}}/N\right) \qquad (1)$$

is the mapping function of variable τ_s.
The variable τ_s used here to detect the smoothness of the solution is

$$\tau_s = \begin{cases} 0, & \text{if } \tau_5^\eta < O(\Delta x^2) \\ \tau_5^\eta, & \text{else.} \end{cases} \qquad (2)$$

The parameters $\delta = 10$, $\gamma = 1$, $\kappa_{\min} = 0.1$ and $\eta_{\max} = \max(\eta_0, \eta_1, \eta_2)$ are used.

5.1. 1D Numerical Results

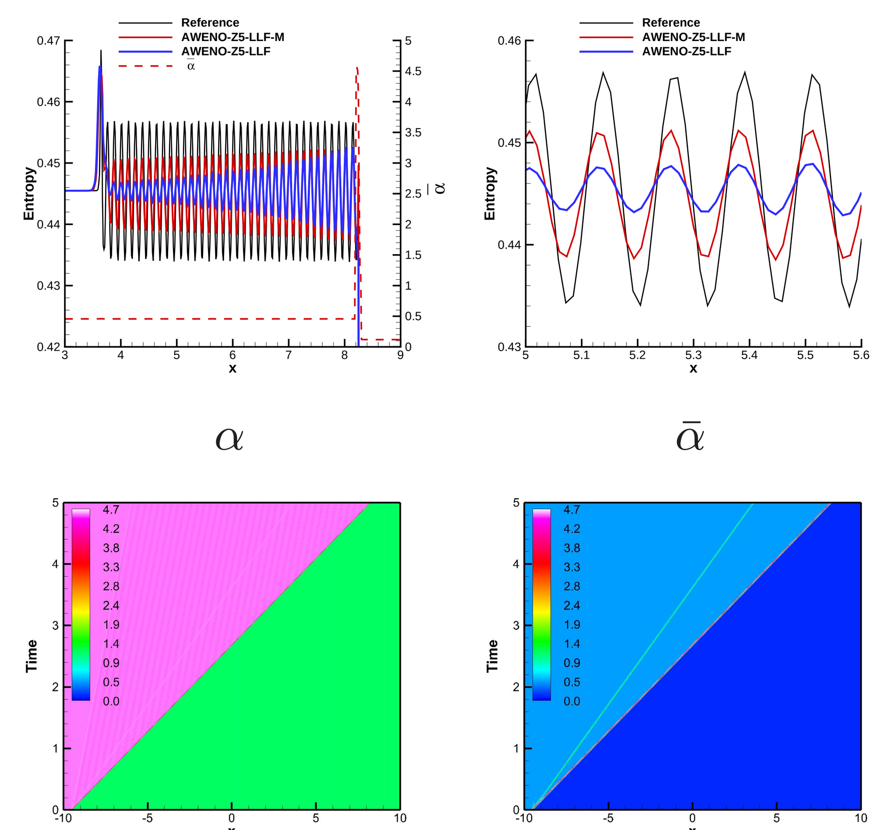

Figure 2: Entropy(Top) and time evolution of numerical diffusion coefficients α and $\bar{\alpha}$(Bottom) of the one-dimensional shock-entropy wave interaction problem computed by the AWENO-Z5 scheme using the LLF and LLF-M fluxes with $N = 1500$.

5.2. 2D Numerical Results

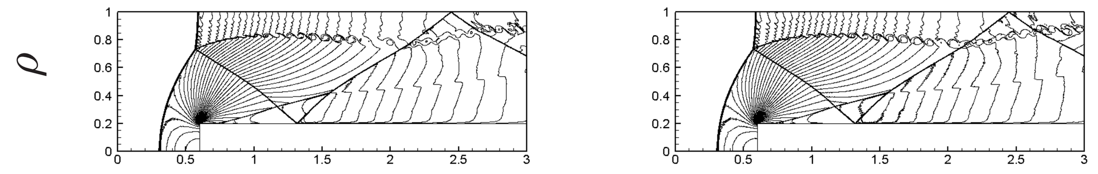

Figure 3: (Color online) The density of FFS problem as computed by the AWENO-Z5 scheme using the LLF and LLF-M fluxes with the mesh resolution $N \times M = 1200 \times 600$ at time $t = 4.0$.

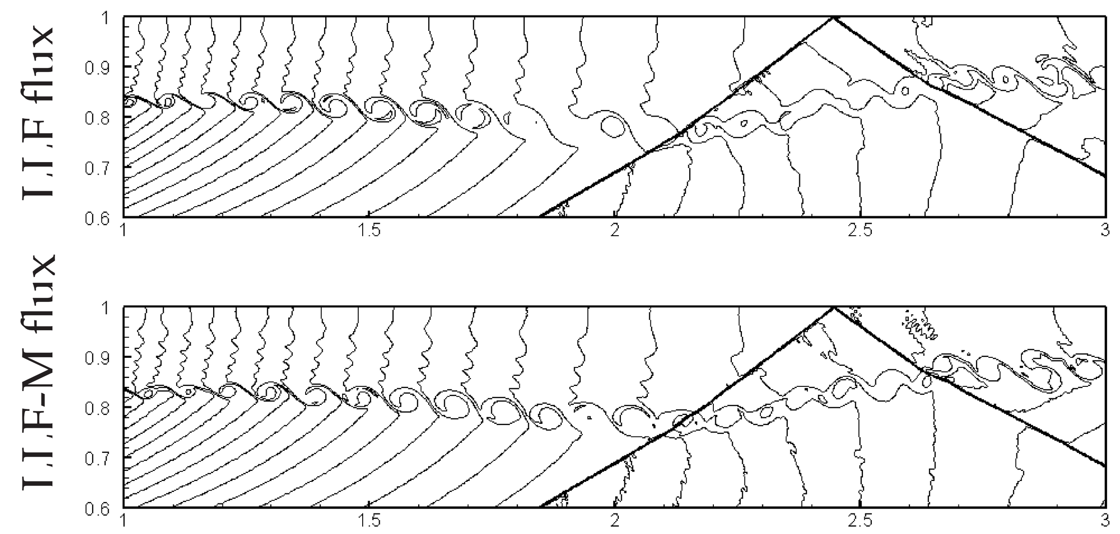

Figure 4: The density of FFS problem.

5.3. The CPU Timings

Table 1: The CPU times (in seconds) of the Riemann problem, the DMR problem and the FFS problem as computed by the AWENO scheme using the LLF and LLF-M fluxes where *(ratio)* are the ratios of CPU times between the proposed flux and the LLF flux.

	$N \times M$	LF flux(s)	LF-M flux(s)
Riemann	400×400	4.5E+03	4.6E+03 (1.02)
FFS	1200×600	2.0E+05	2.2E+05 (1.10)
DMR	800×200	3.4E+03	3.6E+03 (1.05)

The timing shows that the calculation efficiency of the two scheme is basically the same.

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