# Concentration of mass in the pressureless limit of Euler equations for power law

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October. 15, 2019

# Talk Overview

# Introduction

It is known that for general initial data, the solutions to hyperbolic conservation laws develop singularity in finite time, such as the shock discontinuity. And the delta waves is a kind of more singular solutions to the degenerate hyperbolic system like zero pressure gas dynamics.

In this work, the pressureless limit of Riemann solutions to the Euler equation for compressible fluids in power law is studied as the adiabatic exponent goes to zero.

# Introduction

The limit solution, which is a weighted Dirac measure, is proved to be the weak solution for zero pressure Euler equations. Compared with the previous works in the pressureless limit problem, the concave pressure model is considered in this thesis, which has different properties. Follow this idea, we found the pressure function can be generalized to a wider models that possesses some necessary conditions.

# Problem

In this work, the main focus is on the Euler equations of power law in Eulerian coordinates,

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \left(\rho u^2 + p(\rho)\right)_x = 0, \end{cases} \tag{1}$$

with the Riemann initial conditions,

$$(\rho, u)|_{t=0} = \begin{cases} (\rho_-, u_-), & x < 0, \\ (\rho_+, u_+), & x > 0, \end{cases} \tag{2}$$

where $\rho$, $\rho u$ and $p$ denote the density, the momentum, and the scalar pressure, respectively. We assume that $u_- > u_+$. The pressure function

$$p(\rho) = \rho^\gamma, \quad \gamma \in (0, 1). \tag{3}$$

# Problem

System (1)-(2) is just like a hyperbolic system for conservation laws of the form

$$\partial_t V + \partial_x F(V) = 0, \tag{4}$$

with

$$V = \begin{pmatrix} \rho \\ \rho u \end{pmatrix}, \quad F(V) = \begin{pmatrix} \rho u \\ \rho u^2 + p(\rho) \end{pmatrix}.$$

When $\gamma$ goes to zero, the limiting system of (1) formally becomes the zero pressure gas dynamics,

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \left(\rho u^2\right)_x = 0. \end{cases} \tag{5}$$

# Problem

**Riemann solutions to the Euler equation:**

Let's consider the Riemann problem (1), (2) with $u_- > u_+$. The eigenvalues of the system (1) are

$$\lambda_1(\rho, u) = u - \sqrt{p'(\rho)}, \qquad \lambda_2(\rho, u) = u + \sqrt{p'(\rho)}, \tag{6}$$

where $p'(\rho) = \gamma\rho^{\gamma-1}$, $0 < \gamma < 1$. We consider the piecewise smooth solution with a bounded jump along the line $S = \{(\sigma t, t) : 0 < t < \infty\}$. Then by the Rankine-Hugoniot conditions for discontinuity of the system (1), we have

$$\begin{aligned} \sigma[\rho] &= [\rho u], \\ \sigma[\rho u] &= [\rho u^2 + p(\rho)], \end{aligned} \tag{7}$$

from which we can get

$$\sigma = \frac{[\rho u]}{[\rho]},$$

$$u = u_- \pm \sqrt{\left(\frac{1}{\rho_-} - \frac{1}{\rho}\right)(p(\rho) - p(\rho_-))},$$

# Riemann solutions to the Euler equation:

where $\sigma$, $(\rho_-, u_-)$ and $(\rho, u)$ represent the shock speed, the left and right states, respectively

*1-shock curve* $S_1(\rho_-, u_-)$ :

The Lax entropy inequality implies

$$\lambda_1(\rho, u) < \sigma < \lambda_1(\rho_-, u_-),$$

and

$$\sigma - u_- < -\sqrt{p'(\rho_-)} < 0. \tag{8}$$

Then by Rankine-Hugoniot condition, we have

$$(\sigma - u_-)(\rho - \rho_-) = \rho(u - u_-),$$

which indicates that $\rho - \rho_-$ and $u - u_-$ have different signs.

# Riemann solutions to the Euler equation:

If $u > u_-$, then $\rho < \rho_-$ and

$$\sigma - u_- = \frac{\rho}{\rho - \rho_-}(u - u_-) = -\sqrt{\frac{\rho}{\rho_-}}\sqrt{\frac{p(\rho_-) - p(\rho)}{\rho_- - \rho}} = -\sqrt{\frac{\rho}{\rho_-}}\sqrt{p'(\bar{\rho})},$$

for some $\bar{\rho} \in (\rho, \rho_-)$. From a direct computation, we get

$$\sqrt{p'(\rho_-)} - \sqrt{\frac{\rho}{\rho_-}}\sqrt{p'(\bar{\rho})} = \frac{\sqrt{\gamma\rho_-^\gamma} - \sqrt{\gamma\rho\bar{\rho}^{\gamma-1}}}{\sqrt{\rho_-}} > \frac{\sqrt{\gamma\rho_-^\gamma} - \sqrt{\gamma\rho^\gamma}}{\sqrt{\rho_-}} > 0,$$

which gives that

$$\sigma - u_- > -\sqrt{p'(\rho_-)}. \tag{9}$$

This contradicts to (8). Hence we get the first family of shock curve $S_1(\rho_-, u_-)$ in the phase plane, see on the Figure

$$u = u_- - \sqrt{(\frac{1}{\rho_-} - \frac{1}{\rho})(p(\rho) - p(\rho_-))}, \qquad \rho > \rho_-, \; u < u_-. \tag{10}$$

# Riemann solutions to the Euler equation:

*2-shock curve* $S_2(\rho_-, u_-)$,

Similarly we can get the second family of shock curve

$$u = u_- - \sqrt{(\frac{1}{\rho_-} - \frac{1}{\rho})(p(\rho) - p(\rho_-))}, \qquad \rho < \rho_-, \; u < u_-. \tag{11}$$

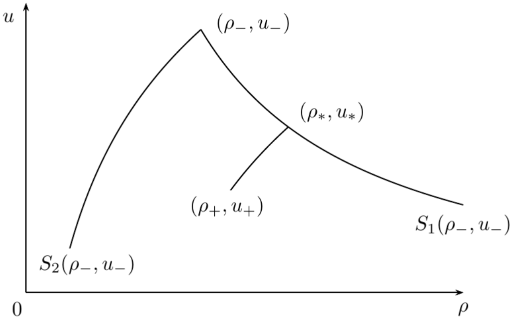


Figure: The shock curves in the phase plane.

# Riemann solutions to the Euler equation:

We are in particular interested in the case $S_1 + S_2$ that their exist a unique intermediate state $(\rho_*, u_*)$ such that $(\rho_*, u_*) \in S_1(\rho_-, u_-)$ and $(\rho_+, u_+) \in S_2(\rho_*, u_*)$, i.e.

$$u_* = u_- - \sqrt{\left(\frac{1}{\rho_-} - \frac{1}{\rho_*}\right)(p(\rho_*) - p(\rho_-))}, \qquad \rho_* > \rho_-, \; u_* < u_-, \tag{12}$$

$$u_+ = u_* - \sqrt{\left(\frac{1}{\rho_*} - \frac{1}{\rho_+}\right)(p(\rho_+) - p(\rho_*))}, \qquad \rho_+ < \rho_*, \; u_+ < u_*, \tag{13}$$

with the shock speed,

$$\sigma_1 = \frac{\rho_* u_* - \rho_- u_-}{\rho_* - \rho_-}, \qquad \sigma_2 = \frac{\rho_+ u_+ - \rho_* u_*}{\rho_+ - \rho_*}, \tag{14}$$

respectively. In this case, the Riemann solution is

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, u_-), & x < \sigma_1 t, \\ (\rho_*, u_*), & \sigma_1 t < x < \sigma_2 t, \\ (\rho_+, u_+), & x > \sigma_2 t. \end{cases} \tag{15}$$

Note that the two shock curves (10), (11) become very close to the line $u = u_-$ as $\gamma$ tends to zero, then for any Riemann data with $u_- > u_+$, the Riemann solutions will contain two shocks as long as $\gamma > 0$ small enough, see Figure 3.1. Hence we only need to consider the case $S_1 + S_2$ as $\gamma$ tends to zero. By (12) and (13), we can get an identity of $\rho_*$,

$$
\begin{aligned}
u_- - u_+ &= \sqrt{\left(\frac{1}{\rho_-} - \frac{1}{\rho_*}\right)(p(\rho_*) - p(\rho_-))} + \sqrt{\left(\frac{1}{\rho_*} - \frac{1}{\rho_+}\right)(p(\rho_+) - p(\rho_*))} \\
&= \sqrt{\frac{{\rho_*}^\gamma}{\rho_-} - {\rho_*}^{\gamma-1} + \frac{{\rho_-}^\gamma}{\rho_*} - {\rho_-}^{\gamma-1}} + \sqrt{\frac{{\rho_*}^\gamma}{\rho_+} - {\rho_*}^{\gamma-1} + \frac{{\rho_+}^\gamma}{\rho_*} - {\rho_+}^{\gamma-1}}.
\end{aligned}
\tag{16}
$$

Thus we get

# The limiting behavior of Riemann solutions

**Lemma 1**

$\lim_{\gamma\to 0} \rho_* = \infty.$

Proof: Set $\liminf_{\gamma\to 0} \rho_* = \alpha$, and $\limsup_{\gamma\to 0} \rho_* = \beta$. If $\alpha < \beta$, then by the continuity of $\rho_*(\gamma)$, there exists a sequence $\{\gamma_n : n \geq 1\} \subseteq (0, 1)$ such that

$$\lim_{n\to\infty} \gamma_n = 0, \quad \text{and} \quad \lim_{n\to\infty} \rho_*(\gamma_n) = c,$$

for some $c \in (\alpha, \beta)$. Then substituting the sequence into the right hand side of (16), and taking the limit we get

$$\lim_{n\to\infty}\left(\frac{\rho_*(\gamma_n)^{\gamma_n}}{\rho_\pm} - \rho_*(\gamma_n)^{\gamma_n-1} + \frac{{\rho_\pm}^{\gamma_n}}{\rho_*(\gamma_n)} - {\rho_\pm}^{\gamma_n-1}\right) = \frac{1}{\rho_\pm} - c^{-1} + \frac{1}{c} - {\rho_\pm}^{-1} = 0,$$

which leads to a contradiction from (16) with the assumption $u_- > u_+$. Hence $\alpha = \beta$.

If $\alpha = \beta \in (0, \infty)$, then $\lim\limits_{\gamma\to 0} \rho_*(\gamma) = \alpha$. We can also get a contradiction when taking limit in (16). So $\alpha = \beta = 0$ or $\alpha = \beta = \infty$. However, the entropy condition shows that $\rho_* > \max\{\rho_-, \rho_+\}$, which leads to $\lim\limits_{\gamma\to 0} \rho_*(\gamma) = \alpha = \beta = \infty$.

# The limiting behavior of Riemann solutions

## Lemma 2

$\lim\limits_{\gamma\to 0} \rho_*^\gamma =: a = 1 + \left(\frac{\sqrt{\rho_+\rho_-}(u_- - u_+)}{\sqrt{\rho_+}+\sqrt{\rho_-}}\right)^2.$

Proof: Now taking limit at right hand side in (16), we have

$$\lim_{\gamma\to 0}\left(\frac{{\rho_*}^\gamma}{\rho_\pm} - {\rho_*}^{\gamma-1} + \frac{{\rho_\pm}^\gamma}{\rho_*} - {\rho_\pm}^{\gamma-1}\right) = \lim_{\gamma\to 0}\frac{\rho_* - \rho_\pm}{\rho_\pm\rho_*}\left({\rho_*}^\gamma - {\rho_\pm}^\gamma\right) =: \frac{a-1}{\rho_\pm},$$

and

$$u_- - u_+ = \sqrt{\frac{a-1}{\rho_-}} + \sqrt{\frac{a-1}{\rho_+}},$$

from where we can get $a = 1 + \left(\frac{\sqrt{\rho_+\rho_-}(u_- - u_+)}{\sqrt{\rho_+}+\sqrt{\rho_-}}\right)^2$.

# The limiting behavior of Riemann solutions

## Proposition 1

*It holds that*

$$\lim_{\epsilon\to 0} u_* = \lim_{\epsilon\to 0} \sigma_1 = \lim_{\epsilon\to 0} \sigma_2 = \sigma, \tag{17}$$

*and*

$$\lim_{\epsilon\to 0} \rho_*(\sigma_2 - \sigma_1) = \sigma[\rho] - [\rho u], \tag{18}$$

*where* $\sigma = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-}+\sqrt{\rho_+}}$.

# The limiting behavior of Riemann solutions

Proof: With a direct computation, we have

$$\lim_{\gamma\to 0} u_* = u_- - \lim_{\gamma\to 0}\sqrt{\frac{{\rho_*}^{\gamma}}{\rho_-} - {\rho_*}^{\gamma-1} + \frac{{\rho_-}^{\gamma}}{\rho_*} - {\rho_-}^{\gamma-1}} = u_- - \sqrt{\frac{a-1}{\rho_-}} = \sigma,$$

$$\lim_{\gamma\to 0}\sigma_1 = \lim_{\gamma\to 0}\frac{\rho_* u_* - \rho_- u_-}{\rho_* - \rho_-} = u_- + \lim_{\gamma\to 0}\frac{\rho_*}{\rho_- - \rho_*}(u_- - u_*) = \sigma,$$

$$\lim_{\gamma\to 0}\sigma_2 = \lim_{\gamma\to 0}\frac{\rho_+ u_+ - \rho_* u_*}{\rho_+ - \rho_*} = u_+ + \lim_{\gamma\to 0}\frac{\rho_*}{\rho_+ - \rho_*}(u_+ - u_*) = \sigma,$$

and

$$\begin{aligned}\lim_{\gamma\to 0}\rho_*(\sigma_2 - \sigma_1) &= \lim_{\gamma\to 0}\rho_*\left(\frac{\rho_+ u_+ - \rho_* u_*}{\rho_+ - \rho_*} - \frac{\rho_* u_* - \rho_- u_-}{\rho_* - \rho_-}\right)\\ &= \lim_{\gamma\to 0}\rho_*\left(\frac{[\rho u]}{\rho_+ - \rho_*} + \frac{-\rho_- u_-[\rho] + \rho_* u_*[\rho]}{(\rho_* - \rho_-)(\rho_* - \rho_+)}\right)\\ &= \sigma[\rho] - [\rho u].\end{aligned}$$

# Weighted of delta wave

## Theorem 2 (Our results)

*Let* $(\rho_\gamma(x,t), m_\gamma(x,t)) = (\rho_\gamma(x,t), \rho_\gamma(x,t)u_\gamma(x,t))$ *be a solution consisting of two shocks for the Riemann problem (1), (2) under the assumption* $u_- > u_+$*. When* $\gamma$ *tends to zero,* $(\rho_\gamma(x,t), m_\gamma(x,t))$ *will converges to*

$$(\rho(x,t), m(x,t)) = (\rho_0(x,t) + w_1(t)\delta_S, \rho_0(x,t)u_0(x,t) + w_2(t)\delta_S)$$

*in the sense of distribution. The singular part of the limit function* $\rho(x,t)$ *and* $m(x,t)$ *is a weighted dirac-measure, of which the weight are*

$$\begin{aligned} w_1(t) &= \frac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho] - [\rho u]), \\ \text{and} \qquad w_2(t) &= \frac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho u] - [\rho u^2]), \end{aligned} \tag{19}$$

*respectively.*

# Numerical Results

The numerical simulations are given for different models of pressure respectively to show the phenomena of concentration of mass by using the first order local Lax-Friedrichs scheme.

We are presenting a designated group of representative numerical results by using Euler system (1), with the Riemann initial data (2). A number of iterative numerical trials are executed to guarantee what we demonstrate are not numerical objects. We solve the initial value problem (1), (2), with $p(\rho) = \rho^\gamma$.

The initial condition is

$$(\rho, u)(x, 0) = \begin{cases} (1.0, 1.5), & \text{for } x < 0, \\ (0.2, 0.0), & \text{for } x > 0. \end{cases}$$

# Numerical Results

To discretize the system (4) , we use the explicit conservative scheme as follows,

$$V_j^{n+1} = V_j^n - \frac{\triangle t}{\triangle x}(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n),$$

where $\triangle t$, $\triangle x$, and $F_{j+\frac{1}{2}}^n$ represent the time step, the spatial size, and the numerical flux, respectively. For the simplicity, we choose the first order local Lax-Friedrichs scheme, which is given below,

$$F_{j+\frac{1}{2}}^n = \frac{1}{2}(F(V_j^n) + F(V_{j+1}^n) - \alpha(V_{j+1}^n - V_j^n)),$$

and

$$\alpha = \max\{\lambda_{\max}(V_j^n), \lambda_{\max}(V_{j+1}^n)\}, \quad \lambda_{\max}(V) = \max\{u - c, u + c\},$$

# Numerical Results

where $c = \sqrt{p'(\rho)}$ represents the sound speed. The time step is obtained by the stability condition namely, $\triangle t = \text{CFL} \cdot \frac{\triangle x}{\max_i |\lambda_{\max}(V_i)|}$, CFL stands for the Courant-Friedrichs-Lewyor(CFL) number, which is taken as 1 here, and 200 points are used.

The numerical simulations with various selection of $\gamma$ (i.e. $\gamma = 1.4, 0.1, 0.05, 0.001$, and the time $t = 0.2$) are demonstrated in Figure.

# Numerical Results

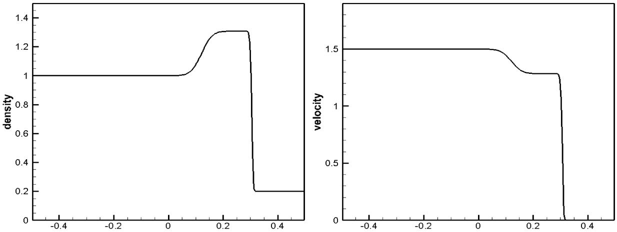


Figure: Density(left) and velocity(right) for $\gamma = 1.4$

# Numerical Results

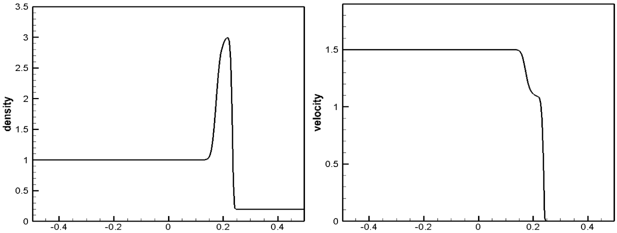


Figure: Density(left) and velocity(right) for $\gamma = 0.1$

# Numerical Results

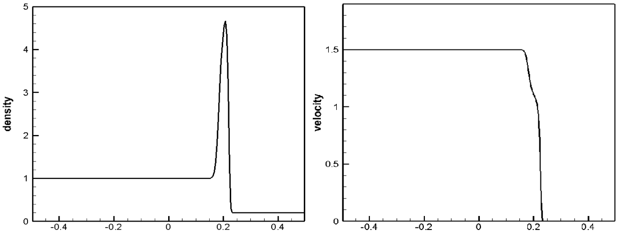


Figure: Density(left) and velocity(right) for $\gamma = 0.05$

# Numerical Results

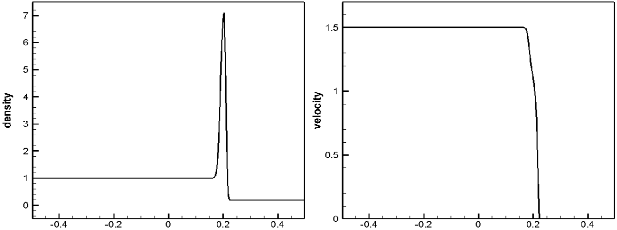


Figure: Density(left) and velocity(right) for $\gamma = 0.001$

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Our results are inspired by


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# Thank you!